A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

IN FOUR BOOKS.

TO WHICH IS ADDED,

A TREATISE ON THE PRIMARY PROPERTIES

OF

CONCHOIDS, THE CISSOID, THE QUADRATRIX, CYCLOIDS, THE LOGARITHMIC CURVE,

AND THE

LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

BY THE

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TO THE REVEREND

CYRIL JACKSON, D. D. F. R. S.

DEAN OF CHRIST CHURCH,

EQUALLY EMINENT FOR HIS OWN ABILITIES AND LEARNING,

AND FOR

HIS UNIFORM ENCOURAGEMENT AND PROMOTION

OF TALENTS AND ACQUIREMENTS IN OTHERS,

AS A

TESTIMONY OF THE HIGHEST ESTEEM FOR HIS CHARACTER,

AND AS A

TRIBUTE OF GRATITUDE FOR MANY IMPORTANT FAVOURS,

THIS WORK

IS MOST RESPECTFULLY INSCRIBED,

BY HIS MUCH OBLIGED AND FAITHFUL SERVANT,

THE AUTHOR.

ADVERTISEMENT.

THERE are two points of which it seems necessary that the reader of the following Treatise of Conic Sections should be apprized; first, what previous knowledge will be expected from him; and secondly, what extent of information the Treatise itself is intended to afford. The first of these will prevent the young student from entering upon the work till he is duly prepared, and the second will enable him to judge how far it is likely to contribute to the attainments which he has in view.

It is expected then, that the young student should understand thoroughly the first six Books of Euclid, the first twenty-one Propositions of the eleventh Book, the two first of the twelfth, and the first principles of Algebra, and Plane Trigonometry.

As no number can be assigned as a limit to the properties of the conic sections, any treatise on the subject can be supposed to contain only a selection of those which are most important and most useful, either generally, or with reference to the particular design of the Writer. In the present instance the design has been, to furnish the young Mathematician with

ſuch a ſeries of propoſitions as might prepare him for conſidering ſome of the moſt important truths in ſcience, and enable him to enter on the ſtudy of natural philoſophy, with the proſpect of obtaining a thorough knowledge of the ſubject. According to theſe views the ſelection of properties and the extent of the work have been regulated; and at the ſame time the arrangement and diviſion of the whole have been made with a deſign of accommodating two deſcriptions of readers. Thoſe who are conſidered as conſtituting the firſt claſs are ſuppoſed to be deſirous of a general but reſpectable portion of knowledge of the ſubject. For the uſe of ſuch a peruſal of the firſt three Books will be found ſufficient, as they contain the properties of the ſections moſt frequently referred to in pure and mixed mathematics. For thoſe who rank under the ſecond, or higher deſcription, a knowledge of all the four Books will be requiſite, as they complete the original deſign of rendering the whole a preparative for the Newtonian Philoſophy. The Author flatters himſelf indeed, that he ſhall be found to have carried his elucidations of the Principia, in the preſent work, conſiderably beyond what have been attempted in other treatiſes of conic ſections.

Something muſt now be added concerning the particular method, which has been adopted in theſe ſheets, of deducing the primary properties of the ſections from the nature of the cone.

It

It is well known, that about the middle of the seventeenth century a difference of opinion took place among mathematicians concerning the proper ſource from which the properties of the conic ſections ſhould be deduced. But notwithſtanding the objections which then began to be made to their deduction from the cone, and which have ſince been continued, it appears to the Author of this work that the difficulties attributed to the deductions from it were not to be imputed to the ſolid itſelf, but that they were occaſioned ſolely by the manner in which the deductions had been made. The early writers did not happen to perceive that the general and extenſive property, expreſſed in the thirteenth Propoſition of the firſt Book of this Treatiſe, could eaſily be obtained from the cone; and, not adverting to this, their deductions from the cone were ſometimes tedious and intricate.*

The above-mentioned property, as far as ſecants are concerned, occurs (I believe for the firſt time) in a folio volume, of which a treatiſe of conic ſections makes a part, entitled, Euclides Adauctus et Methodicus, &c. *publiſhed by Guarinus in* 1671. *The property to the ſame extent is to be found in Jones's* Synopſis Palmariorum Matheſeos, *publiſhed in* 1706; *but neither of theſe two authors conſidered the property as a fundamental one, nor do they ſeem*

* For foundations for ſyſtems, independent of the cone, ſee the Scholium in page 110, and the firſt ſeven articles in the Scholium at the end of the third Book.

to

to have been aware of the advantages it was capable of producing. Its extensive utility was first evinced in Hamilton's Conic Sections, published in Latin in 1758; *and on the appearance of this work objections to the cone ought to have ceased.*

This was my persuasion when I published my former treatise *, *and every deliberation on the subject since has tended to strengthen my conviction of its justice for the following reasons. First, the whole trouble with the cone is reduced to a very few demonstrations, for which no farther knowledge of Euclid is necessary than what is requisite for Spherical Trigonometry. Secondly, by this method the general properties are obtained with most ease and elegance. Lastly, by deducing the properties from the cone the treatise is rendered more extensively useful. No work on conic sections, confined to their description on a plane, can be applied to elucidations in Perspective, Projections of the Sphere, the Doctrine of Eclipses, and in some other particulars of the highest importance in science.*

For the rest it need only be said, that the manner, in which the properties of the sections are classed and arranged, appeared to the Author, on the whole,

* In the year 1792 the author of the present work published a quarto volume, entitled, *Sectionum Conicarum Libri septem. Accedit Tractatus de Sectionibus Conicis, et de Scriptoribus qui earum doctrinam tradiderunt.* The last mentioned Tract contains a full historical account of the subject.

to

to be that which was beſt calculated to ſhew what properties are general, and what are appropriate to each of the ſections.

The treatiſe following that on conic ſections, in the preſent volume, contains only the moſt common properties of the curves ſpecified in the title page. It is intended as a preparative for thoſe who wiſh to inveſtigate the higher properties by means of Fluxions. In the firſt ſection methods of finding two mean proportionals and triſecting an angle, by means of conchoids, are inſerted. In the third ſection a method of dividing an angle in any given proportion, by means of the quadratrix, is given; as is alſo the quadrature of the circle, by means of the ſame curve.

N. B. When a demonſtration, in the following work, is effected by means of two ranks of magnitudes, which, taken two and two in the ſame or in a croſs order, have the ſame ratio to one another, they are placed thus,*

A : B : C : D

E : F : G : H;

A, B, C, D *repreſenting the firſt rank, and* E, F, G, H *the ſecond. Previous to this arrangement of the magnitudes, their ratio to one another is eſtabliſhed, and therefore it evidently appears in which of the two orders the magnitudes are proportional. In*

* That is either ex æquali, or ex æquali in proportione perturbata.

order

of contraction, $\left.\begin{matrix} t.\,AB^{2} \\ \text{or} \\ ſ.\,AB^{1} \end{matrix}\right\}$, *which means the ſquare of* AB *if a tangent, or the rectangle under its ſegments if a ſecant.*

ERRATA.

Page 14. in the Corollary *read* can cut a ſcalene cone

— 128. line 4. from the bottom, *for* it is a ſecond diameter *read* C S is a ſecond diameter

— 226. line 9. *read* Nicomedes

— 240. line 21. *read* Dinoſtratus

— 247. line 7. *for* B D *read* E D

— 247. in the margin *read* Fig. 13.

— 258. line 16, &c. *read* analogous to a ſeries of logarithms, and the ordinates B F, C G, D H, &c. are analogous to the natural numbers of theſe logarithms.

— 259. line 15. and 18. *for* curve *read* ſpiral

LEMMAS

LEMMAS

FOR THE

FIRST THREE FOLLOWING BOOKS

OF

CONIC SECTIONS.

LEMMA I.

If the plane figure A F D *be bounded by the straight line* A D *and the curve* A F D, *and if the square of the straight line* F I, *drawn from any point* F *in the curve perpendicular to* A D, *be equal to the rectangle under the segments* A I, I D, *the figure will be a semicircle.* Fig. 1.

For let A D be bisected in C, and draw C F. Then (47. i.) the square of C F is equal to the squares of C I, I F together, and therefore, by hypothesis, equal to the square of C I together with the rectangle under A I, I D. But as A D is bisected in C, the square of A C or C D (5. ii.) is equal to the square of C I together with the rectangle under A I, I D. Consequently the square of C F is equal to the square of A C or C D; and therefore C F is equal to A C or C D. The figure A F D is therefore a semicircle. For Prop. IV. Book I.

If two straight lines be parallel, and a plane pass through each of them, the common section of these planes, if they cut one another, will be parallel to each of the parallel lines.

Fig. 2. Let two straight lines as A D, B C be parallel. Let the plane A D F pass through A D, and cut the plane B C F, passing through B C, in the straight line E F; the line of common section E F is parallel to A D, B C.

For let these planes be cut by two parallel planes E A B, F D C. Let the plane E A B cut the plane A B C D in the straight line A B, the plane A D F E in A E, and the plane B C F E in B E. Let the plane F D C cut the plane A B C D in the straight line D C, the plane A D F E in D F, and the plane B C F E in C F. Then the straight line A B is parallel to D C (16. xi.) A E is parallel to D F, and B E is parallel to C F. Hence (34. i.) A B, D C are equal; the angle E A B (10. xi.) is equal to the angle F D C, and the angle E B A is equal to the angle F C D. Consequently (26. i.) A E, D F are equal, and therefore (33. i.) A D, E F are equal and parallel. For the same reasons E F, B C are parallel. For Prop. VII. Book I.

LEMMA III.

If a straight line cut either of two parallel straight lines, and be in the same plane with them, it will, if sufficiently produced, cut the other.

Fig. 3. Let the straight lines A B, D E be parallel, and let C F be in the same plane with them, and cut A B in the point C; the straight line C F produced will cut D E.

For let any point G be taken in D E, and draw C G. Then as the angles B C G, E G C together (29. i.) are equal to two right angles, the angles F C G, E G C together

gether are lefs than two right angles, and therefore (ax. 12. i.) the ftraight lines C F, D E produced will meet one another. For Prop. VIII. Book I.

LEMMA IV.

If two ftraight lines cutting one another be parallel to a plane, a plane paffing through them will be parallel to the fame plane.

Let the two ftraight lines A B, C B, cutting one another in B, be parallel to the plane D G H E; the plane paffing through A B, C B is parallel to the plane D G H E. Fig. 4.

For let F be any point in the plane D G H E. Through A B and F let a plane be paffed, and let it cut the plane D G H E in the ftraight line D F H; and let a plane paffing through C B and F cut it in E F G. Then will A B be parallel to D F H, and C B will be parallel to E F G. For if not, then A B will meet D F H, and C B will meet E F G, and confequently A B, C B will meet the plane D G H E, in which D H, E G are, contrary to the hypothefis. The plane paffing through A B, C B (15.xi.) is therefore parallel to the plane D G H E. For Prop. X. Book I.

LEMMA V.

If the firft of eight ftraight lines be to the fecond as the third to the fourth, and if the fifth be to the fixth as the feventh to the eighth; then the rectangle under the firft and fifth will be to the rectangle under the fecond and fixth as the rectangle under the third and feventh to the rectangle under the fourth and eighth. And if the rectangle under the firft and fifth, of eight ftraight lines, be to the rectangle under the fecond and fixth as the rectangle under the third and feventh to the rectangle under the fourth and eighth, and if the firft be to the fecond as

 the

the third to the fourth, then the fifth will be to the sixth as the seventh to the eighth.

Fig. 5. Part I. Let A B, the first of eight straight lines, be to B C the second, as D E the third to E F the fourth, and let G B the fifth be to B H the sixth as I E the seventh to E K the eighth; then the rectangle under A B, G B is to the rectangle under B C, B H as the rectangle under D E, I E to the rectangle under E F, E K.

For let A B, B C be in a straight line; G B, B H be in a straight line; D E, E F be in a straight line; and I E, E K be in a straight line; and let these straight lines be at right angles to one another, and let the rectangles be completed as represented in the figure. Then A G is the rectangle under A B, G B; C H is the rectangle under B C, B H; D I is the rectangle under D E, I E; and F K is the rectangle under E F, E K. By hypothesis A B : B C :: D E : E F, and therefore (11. v. and 1. vi.) A G : G C :: D I : I F. Again, by hypothesis, G B : B H :: I E : E K, and therefore (11. v. and 1. vi.) G C : C H :: I F : F K. Consequently,

A G : G C : C H
D I : I F : F K,

and therefore (22. v.) A G : C H :: D I : F K.

Fig. 5. Part II. The construction, with respect to the rectangles, remaining as stated above, let the rectangle A G be to the rectangle C H as the rectangle D I to the rectangle F K, and let A B be to B C as D E to E F; then G B is to B H as I E to E K.

For, by hypothesis and inversion, B C : A B :: E F : D E; and therefore (11. v. and 1. vi.) G C : A G :: I F : D I. Again by hypothesis A G : C H :: D I : F K. Consequently,

G C : A G : C H
I F : D I : F K;

and

and therefore (22. v.) G C : C H : : I F : F K. But (1. vi.) G C : C H : : G B : B H, and I F : F K : : I E : E K; and therefore (11. v.) G B : B H : : I E : E K. For Prop. X. Book I.

LEMMA VI.

If the points C, D *be so situated in the straight line* A B, *that the rectangle* D A C *is equal to the rectangle* C B D, *then* A C *is equal to* B D: *or if the rectangle* A C B *be equal to the rectangle* B D A, *then* A C *is equal to* B D. Fig. 6.

Case I. Let C D be bisected in E, and then (6. ii.) the rectangle D A C together with the square of E C is equal to the square of A E; and the rectangle C B D together with the square of E D is equal to the square of B E. The squares of A E, B E are therefore equal, and consequently A E is equal to B E, and A C is equal to B D.

Case II. Let A B be bisected in E, and then (5. ii.) the rectangle A C B together with the square of E C is equal to the square of A E or E B; and the rectangle B D A together with the square of E D is equal to the square of E B. The squares of E C, E D are therefore equal, and consequently E C is equal to E D, and A C is equal to B D. For Prop. I. Book II.

LEMMA VII.

If a straight line touch a circle, and two straight lines cutting the circle pass through the point of contact, and meet a straight line parallel to the tangent, the rectangle under the segments of the one, between the point of contact and circumference, and between the point of contact and straight line parallel to the tangent, will be equal to the rectangle under the segments of the other, between the point of contact and circumference and between the point of contact and straight line parallel to the tangent.

Fig. 7. and 8. Let the ſtraight line A B be parallel to the ſtraight line R G touching the circle E P F in the point P, and let the two ſtraight lines E P, F P, paſſing through P, meet the circumference again, the one in E and the other in F, and let E P meet the ſtraight line A B in C, and F P meet it in D; then the rectangle under E P, P C is equal to the rectangle under F P, P D.

For E F being drawn, the angle E F P (32. iii.) is equal to the angle R P E, which (29. i.) is equal to the angle P C D. The triangles E P F, D P C are therefore equiangular, and (4. vi.) E P : P F :: D P : P C. Conſequently, (16. vi.) the rectangle under E P, P C is equal to the rectangle under F P, P D*. For Prop. I. Book III.

Cor. 1. If A B cut the circle in B, and P B be drawn, it may be proved in the ſame way that the rectangle under F P, P D is equal to the ſquare of P B. For the tangent R P being produced to G, and B F being drawn, the angle B P G (32. iii.) is equal to the angle B F P; and (29. i.) it is alſo equal to the angle D B P. The triangles B F P, D B P are therefore equiangular, and (4. vi.) F P : P B :: P B : P D, and (17. vi.) the rectangle under F P, P D is equal to the ſquare of P B.

Fig. 8. *Cor.* 2. The reſt remaining as above, if B G be drawn parallel to F P, B G is (34. i.) equal to P D, and therefore by the above $FP = \frac{PB^2}{BG}$.

LEMMA VIII.

If the firſt of three ſtraight lines be to the third as the ſquare of the ſum of the firſt and ſecond to the ſquare of the ſum of the ſecond and third, the ſecond will be a

* The ſtraight line A B may be on either ſide of the tangent R G, and it is not neceſſary, upon being produced indefinitely, that it ſhould meet the circumference of the circle.

mean

first and second to the square of the difference of the second and third, the second will be a mean proportional between the first and third.

Part I. Let A denote the first, B the second, and C the third, of the straight lines. Then by hypothesis $A : C :: \overline{A+B}^2 : \overline{B+C}^2$; and it is to be proved that B is a mean proportional between A and C.

Let D be a mean proportional between A and C, and then, by inversion, $D : A :: C : D$, and (18. v.) $A+D : A :: D+C : D$; and therefore by alternation $A : D :: A+D : D+C$. Consequently (22. vi.) $A^2 : D^2 :: \overline{A+D}^2 : \overline{D+C}^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$, and therefore by hypothesis (and 11. v.) $\overline{A+B}^2 : \overline{B+C}^2 :: \overline{A+D}^2 : \overline{D+C}^2$. Consequently (22. vi.) $A+B : B+C :: A+D : D+C$; and therefore, by conversion, $A+B : A-C :: A+D : A-C$, and (14. v.) $A+B$ is equal to $A+D$. Consequently B is equal to D, and therefore B is a mean proportional between A and C.

Part II. Let A denote the first of the three straight lines, B the second, and C the third. Then by hypothesis $A : C :: \overline{A-B}^2 : \overline{B-C}^2$; and it is to be proved that B is a mean proportional between A and C.

Let D be a mean proportional between A and C. Then $A : D :: D : C$, and by conversion $A : A-D :: D : D-C$; and by alternation, $A : D :: A-D : D-C$. Consequently (22. vi.) $A^2 : D^2 :: \overline{A-D}^2 : \overline{D-C}^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$; and therefore by hypothesis (and 11. v.) $\overline{A-B}^2 : \overline{B-C}^2 :: \overline{A-D}^2 : \overline{D-C}^2$. Consequently (22. vi.) $A-B : B-C :: A-D : D-C$; and therefore (18. v.) $A-C : B-C :: A-C : D-C$. Hence (14. v.) $B-C$ is equal to

D − C, and therefore B is equal to D. Confequently B is a mean proportional between A and C. For Prop. XXIII. Book III.

A GEOMETRICAL TREATISE OF *CONIC SECTIONS.*

BOOK I.

Containing general Properties deduced from the Cone.

DEFINITIONS.

I.

IF through the point V, without the plane of the circle A F B, a ftraight line A V D be drawn, and extended indefinitely both ways, and if the point V remain fixed, and the ftraight line A V D be moved round the whole circumference of the circle, two Superficies will be generated by its motion, each of which is called a *Conical Superficies*; and thefe mentioned together are called *Oppofite Superficies*. Fig. 9.

Cor. A ftraight line drawn from the fixed point V to any point G in either fuperficies is wholly in that fuperficies; and, being produced, the part on the other fide of V is wholly in the oppofite fuperficies. For a ftraight line

superficies; and, being produced, the part beyond V is in the opposite superficies. Hence the Cor. is evident; for only one straight line can be drawn from V to G, as two straight lines cannot inclose a space.

II.

The solid contained by the conical superficies and the circle A F B is called *a Cone.*

III.

The fixed point V is called the *Vertex of the Cone.*

IV.

The circle A F B is called the *Base of the Cone.*

V.

Any straight line drawn through the vertex of the cone to the circumference of the base is called a *Side of the Cone.*

VI.

A straight line V C, drawn through the vertex of the cone and the center of the base, is called the *Axis of the Cone.*

VII.

If the axis of the cone be perpendicular to the base, it is called a *Right Cone.*

VIII.

If the axis of the cone be not perpendicular to the base, it is called a *Scalene Cone.*

IX.

A plane is said to *touch a conical superficies,* when it meets the superficies, and when, being produced indefinitely, in any direction, it falls without the superficies.

X.

A straight line which meets a conical superficies, and which, being produced both ways, falls without the super-

ſuperficies, is called a *Tangent*; but a ſtraight line which meets a conical ſuperficies in two points, or each of the oppoſite ſuperficies in one point, is called a *Secant*.

XI.

A ſtraight line is ſaid to be parallel to a plane, when both being produced ever ſo far, both ways, they do not meet.

XII.

If a cone be cut by a plane, their common interſection is called a *Conic Section*.

XIII.

The common interſection of any plane, not paſſing through the vertex of the cone, with the conical ſuperficies, is called the *Curve of a Conic Section*.

PROP. I.

If a cone be cut by a plane paſſing through the vertex, the ſection will be a triangle.

Let the cone V A F B be cut by a plane paſſing through V the vertex, and let V A B be the common interſection of the cone and plane; the ſection V A B is a triangle. Fig. 9.

For let the plane, paſſing through V, cut the plane of the baſe in the ſtraight line (3. xi.) A B, and the circumference of the baſe in the points A, B; and let the ſtraight lines V A, V B be drawn. Then, as the points V, A, B are in the plane cutting the cone, the ſtraight lines V A, V B are wholly in the ſame plane; and as the points A, B are in the conical ſuperficies, the ſtraight lines V A, V B are alſo wholly in the ſuperficies, by the corollary to the firſt Definition. The ſtraight lines V A, V B are therefore the common interſections of the conical ſuperficies and the plane cutting the cone; and conſequently the ſection V A B is a triangle.

Cor.

BOOK I.

Cor. If a plane, passing through the vertex, cut a cone, it will cut the opposite superficies in two straight lines, and only in those two. For if the plane V A B be extended on both sides of the vertex, it will cut the opposite superficies in the straight lines V A, V B produced, and in them only. This is evident from the above, and the corollary to the first Definition.

PROP. II.

If either of the opposite conical superficies be cut by a plane parallel to the base of the cone, the common intersection of the superficies and the plane will be the circumference of a circle, and its center will be in the axis of the cone.

Fig. 10. Let the superficies V A B D, or its opposite superficies, be cut by a plane parallel to the base A B D of the cone, and let F G H be the common intersection of the superficies and this plane; F G H is the circumference of a circle, and its center is in V C, the axis of the cone, or in V C produced.

Let C be the center of the base, and let the axis V C cut the plane F G H in the point I. From the point I, and in the plane F G H, draw any two straight lines I F, I G to the conical superficies. Through V I C, I F let a plane be passed, and let it cut the superficies in the side V F A, and the base of the cone in the straight line C A. Let a plane also be passed through V I C, I G, and let it cut the superficies in the side V G B, and the base of the cone in the straight line C B. Then (16. xi.) F I, A C, and also G I, B C are parallel to one another, each to each: and (29. i.) the triangles A C V, F I V, and also the triangles B C V, G I V are equiangular, each to each. Consequently (4. vi.) A C : F I :: V C : V I, and also V C : V I :: B C : G I; and therefore (11. v.) A C : F I :: B C : G I, and as A C is equal to B C, F I is (14. v.) equal to G I. In the same manner it may be proved

that

2.

Fig. 1.

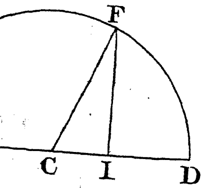


Fig. 2.

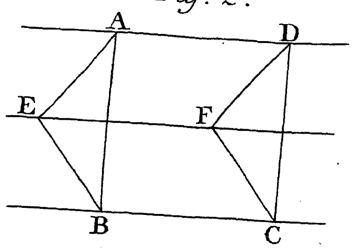


Fig. 3.

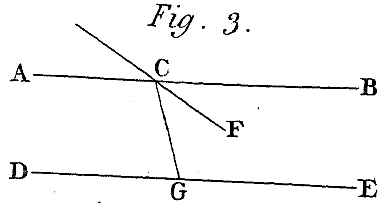


Fig. 4.

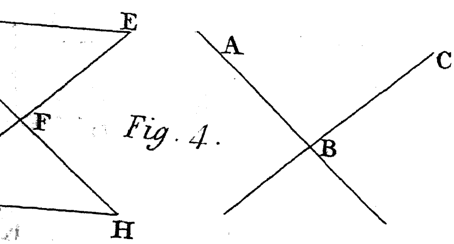


Fig. 5.

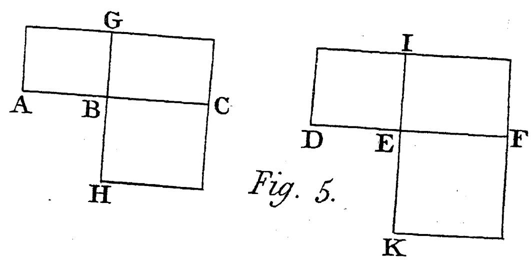


Fig. 6.

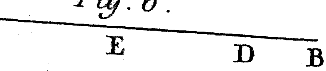


Fig. 7.

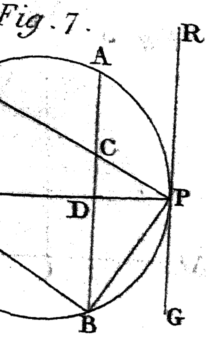


Fig. 8.

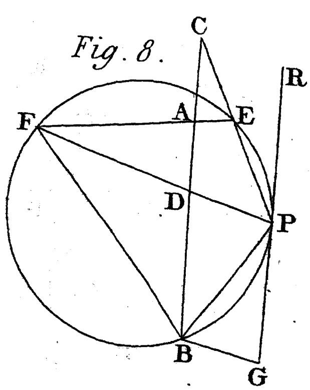
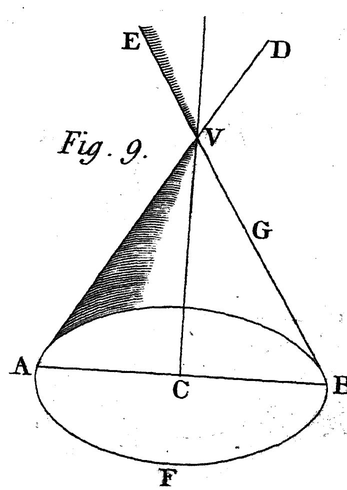


Fig. 9.



that any other ſtraight line drawn in the plane F G H from the point I to the conical ſuperficies is equal to F I; and therefore F G H is the circumference of a circle, and the point I, in the axis V C, is its center. BOOK I.

Cor. 1. From this Propoſition, and the firſt Definition, it appears that any circle parallel to the baſe of the cone, having its center in the axis, and its circumference in either of the oppoſite ſuperficies, may be taken for the baſe of the cone.

Cor. 2. The ſolid contained by the conical ſuperficies V F G H, oppoſite to V A B D, and the circle F G H, is a cone.

PROP. III.

If a ſcalene cone be cut through the axis by a plane perpendicular to the baſe, of the ſides of the ſection, meeting in the vertex, one will be the greateſt, and the other the leaſt of all the ſides of the cone.

For let V N O P be a ſcalene cone, of which V is the vertex, N O P the baſe, and C the centre of the baſe. Let the ſtraight line V B be perpendicular to the plane of the baſe, and meet it in B. Draw the ſtraight line B C, and let it meet the circumference of the baſe in the points P, N. Through the ſtraight lines V B, B C let a plane be paſſed, and let it cut the cone; and let the ſection formed, with the cone, be the triangle V N P, as in the firſt Propoſition. Then as the plane of the triangle V N P paſſes through C, it cuts the cone through the axis*; and, as it paſſes through V B, it is alſo (18. xi.) perpendicular to the Fig. 11. and 12.

* As the repreſentation of the axis could not render the demonſtration more perſpicuous, it was intentionally omitted in the figures. The Reader will find the ſame omiſſion in the figure for Prop. IV. and V. and intentionally made, for the ſame reaſon.

baſe

BOOK I.

base N O P. If therefore the point B be farther from N than from P, it remains to be proved, that V N is greater and V P less than any other side of the cone.

Let V O be any other side of the cone, and let it meet the circumference of the base in O, and draw B O. Then, as V B is perpendicular to the plane of the base, the angles V B N, V B P, V B O are right angles; and therefore (47. i.) the square of V N is equal to the squares of V B, B N together; and the square of V O is equal to the squares of V B, B O together; and the square of V P is equal to the squares of V B, B P together. But (7. and 8. iii.) B N is greater than B O, and B O is greater than B P; and therefore the square of B N is greater than the square of B O, and the square of B O is greater than the square of B P. Consequently the square of V N is greater than the square of V O, and the square of V O is greater than the square of V P. Of all the sides of the cone therefore, V N is the greatest, and V P is the least.

If the perpendicular V B fall into the circumference of the base, then B and P will coincide; and (referring to 15. iii. instead of 7. and 8. iii) the demonstration will be the same as above*.

Cor. As there can be only one perpendicular to a plane (13. xi.) drawn from the same point above the plane, it is evident from the demonstration of this Proposition that only one plane can cut a cone through the axis, and be perpendicular to the base.

* It is evident that all the sides of a right cone are equal to one another. For, in this case, the perpendicular to the base, drawn from V, will fall into C, the centre, according to the seventh definition.

PROP.

BOOK I.

Fig. 13.

PROP. IV.

Let the scalene cone V N O P *be cut by a plane passing through the axis, and perpendicular to the base* N O P, *and let the common section be the triangle* V N P; *in the side* V P *take any point* D, *and in the plane of the triangle make the angle* V D A *equal to the angle* V N P; *then if the cone be cut by a plane passing through* D A, *and perpendicular to the triangle* V N P, *its common section* A F D B *with the cone will be a circle.*

For let the side V P be less than the side V N, as in the preceding Proposition, and produce A D to T and N P to R. Then, as V N is greater than V P, the angle V P N (18. i.) is greater than the angle V N P. But the angles V N P, V D A are equal, by hypothesis; and as the angle P D T is equal (15. i.) to the angle V D A, the angle V P N is greater than the angle P D T. The angles V P N, D P R together are therefore greater than the angles P D T, D P R together; and consequently the angles P D T, D P R together are less than two right angles. If therefore the straight lines A D, N P be sufficiently produced they will meet. Let them be produced and meet in R; and let the plane of the section A F D B cut the plane of the base N O P in the straight line R S. In D A take any point I, and let the cone be cut by a plane passing through I and parallel to the base; and let the section formed be the circle H F K B, as in the second Proposition. Let the circle H F K B cut the triangle V N P in the straight line H I K, and the section A F D B in the straight line F I B. Then as the section H F K B is parallel to the plane of the base, and as these parallel planes are cut by the plane of the section A F D B, the common sections (16. xi.) F I B, S R are parallel; and as the plane of the section A F D B, and the plane of the base N O P are perpendicular to the plane

BOOK I.

plane of the triangle V N P, and cut one another in S R, the straight line S R (19. xi.) is perpendicular to the plane of the triangle V N P. The straight line F I B (8. xi.) is therefore perpendicular to the plane V N P, and consequently (4. xi.) perpendicular to H K, A D. But, as the plane of the triangle V N P passes through the axis, H K is a diameter of the circle H F K B by the second Proposition, and therefore (3. iii.) F B is bisected in I. Consequently (35. iii.) the rectangle under K I, I H is equal to the square of F I or B I. Again, as the circle H F K B is parallel to the plane of the base, and as these parallel planes are cut by the triangle V N P, the common sections (16. xi.) H I K, N P are parallel. The angle A H I (29. i.) is therefore equal to the angle V N P, and consequently equal to the angle K D I. The angles (15. i.) A I H, K I D are also equal, and therefore the triangles A I H, K I D are equiangular. Consequently (4. vi.) A I : I H : : K I : I D, and (16. vi.) the rectangle under A I, I D is equal to the rectangle under K I, I H; and therefore, by the above, the rectangle under A I, I D is equal to the square of F I or I B. The section A F D B is therefore a circle, by the first Lemma.

The circle A F D B formed in a scalene cone, in the manner mentioned in the Proposition, is called *a Subcontrary Section.*

PROP. V.

If a conic section be a circle, and be not parallel to the base of the cone, it will be a subcontrary section.

Fig. 13. Let the cone V N O P be cut by a plane not parallel to the base N O P, and let the section A F D B formed by it, with the cone, be a circle; A F D B is a subcontrary section.

For let I be the point in which the axis of the cone meets

meets the circle A F D B, and through I let a plane be passed parallel to the base, and let F K B H be the circle formed by it with the cone, as in the second Proposition. Let B F be the common section of this circle with the circle A F D B. Then by Prop. II. the point I is the center of the circle F K B H, and consequently B F is bisected in I. Through I draw in the circle A F D B the straight line A D at right angles to B F; and through A D and V, the vertex, let a plane be passed, and let V N P be the triangle formed by it with the cone, as in the first Proposition. Let H K be the line of common section of the triangle V N P and the circle F K B H. Let L be any point in A D, and through L let a plane be passed parallel to the base N O P, or to the circle F K B H. Let M C E G be the circle formed by this plane with the cone, and let M E be its line of common section with the triangle V N P, and C L G its line of common section with the circle A F D B. Then (16. xi.) the straight lines H K, M E are parallel, as are also B I F, C L G; and as A I B is a right angle, A L C is (29. i.) a right angle. Again, as the straight line A D bisects the straight line B F at right angles, A D (Cor. 1. iii.) is the diameter of the circle A F D B. The straight line G C (3. iii.) is therefore bisected in L. But as I is the point in which the axis of the cone meets the circle A F D B, it is evident that the triangle V N P cuts the cone through the axis, and consequently by Prop. II. M E is a diameter of the circle M C E G, and the point L is not its center. Hence the diameter M E (3. iii.) bisects G C in L at right angles, and G L is at right angles to A D, M E, and therefore it is at right angles to the triangle (4. xi.) V N P. Consequently (18. xi.) each of the sections A F D B, M G E C is at right angles to the triangle V N P, and therefore as M G E C is parallel to the base, the cone is cut by the plane

BOOK I. V N P paſſing through the axis of the cone, and perpendicular to the baſe N O P. Again as G L is at right angles to each of the two diameters M E, A D, the rectangle under M L, L E is equal to the rectangle under D L, L A, each of theſe rectangles (35. iii.) being equal to the ſquare of G L; and therefore (16. vi.) D L : L E :: M L : L A, and (6. vi.) the angle L D E is equal to the angle A M L, or (29. i.) V N P. The circle A F D E is therefore a ſubcontrary ſection.

Cor. A conic ſection neither parallel to the baſe of the cone, nor a ſubcontrary ſection, is not a circle.

DEFINITIONS.

XIV.

Fig. 10. The cones V A B D, V F G H, having the common vertex V, and whoſe ſuperficies are oppoſite, being generated by the ſame line as in the firſt Definition, are called *Oppoſite Cones*.

Cor. It is evident from this, and the ſecond Propoſition, that if either of the oppoſite cones be cut by a plane parallel to the baſe of either, the ſection will be a circle.

XV.

Fig. 15. If the plane V B E touch the conical ſuperficies in the ſide V B, and the cone V A B F be cut by the plane F D C parallel to the plane V B E, the ſection F D C, formed by the cutting plane and the cone, is called a *Parabola*.

XVI.

The plane V B E is called the *Vertical Plane to the Parabola*.

Cor. 1. As the cone may be indefinitely extended, it is evident that the parabola may alſo be indefinitely extended; and as the parabola does not ſurround the cone, it is evident that its curve does not include a ſpace.

Cor.

Cor. 2. An indefinite number of ſtraight lines parallel to V B may be drawn in the plane of the parabola. For the common ſection of any plane paſſing through V B, and any point in the parabola, with the parabola (16. xi.) will be parallel to V B.

XVII.

If the cone V A B C be cut by a plane, and if the ſection D K L H, formed by the plane and the cone, ſurround the cone, and is not a circle, it is called an *Ellipſe*. Fig. 16.

XVIII.

If the oppoſite cones V A B E, V M N, be cut by a plane V B E paſſing through the vertex V, and if they be alſo cut by a plane parallel to V B E, forming with the oppoſite cones the ſections F D C, Q R S; each of the ſections F D C, Q R S is called an *Hyperbola*, and when mentioned together they are called *Oppoſite Hyperbolas*. Fig. 17.

XIX.

The plane V B E is called the *Vertical Plane* to the Hyperbola, or Oppoſite Hyperbolas.

Cor. 1. It is evident, as each of the oppoſite cones may be indefinitely extended, that an hyperbola may be indefinitely extended; and that its curve does not include a ſpace.

Cor. 2. An indefinite number of ſtraight lines parallel to V B or V E may be drawn in the plane of the oppoſite hyperbolas. For the common ſection of any plane paſſing through V B, or V E, and any point in either hyperbola, with the plane of the hyperbolas, will be parallel (16. xi.) to V B or V E.

XX.

A ſtraight line in the plane of a conic ſection, which meets the curve, and which being produced both ways falls without it, is called a *Tangent*; but a ſtraight line

BOOK I. which meets the curve of a conic ſection in two points, or each of the oppoſite hyperbolas in one, is called *a Secant.*

SCHOLIUM.

Although only the Parabola, Ellipſe, and Hyperbola, are denominated Conic Sections, the attentive reader will readily perceive from the foregoing Propoſitions and Definitions, that five different Sections may be formed by the interſection of a cone and a plane varying its poſition. For if a ſtraight line parallel to the baſe be within the cone and remain fixed, and a plane move about it as an axis, when the plane paſſes through the vertex, the interſection of the cone and plane will be a triangle, as in the firſt Propoſition. When the plane has moved from the vertex, but ſtill cuts both the oppoſite cones, the ſection formed in each will be an hyperbola, as in the eighteenth Definition. When the plane, proceeding in its motion round the fixed ſtraight line, has arrived at a poſition parallel to that of a plane touching the cone in one of its ſides, the ſection which it then forms with the cone is a parabola, as in the fifteenth Definition. In any other poſition of the moving plane, beſides thoſe already mentioned, an ellipſe or circle will be formed with the cone, according to the circumſtances ſtated in the ſeventeenth Definition, and in the ſecond and fourth Propoſitions.

PROP. VI.

One ſtraight line, and one only, can be drawn to touch a conic ſection in a given point in the curve.

Fig. 14. Let GDH be a conic ſection, and let D be a given point in the curve; through D one ſtraight line, and only one, can be drawn to touch the ſection.

Let

Let V be the vertex of the cone V A C B, and through D draw V C a fide of the cone, meeting the bafe in the point C. Draw C F (17. iii.) touching the bafe, and through V C, C F let a plane pafs, and let its line of common fection with the plane of the fection G D H be D E. Then the ftraight line D E touches the fection G D H, and no other ftraight line can touch it in D.

For, as C F meets the bafe in the point C only, it is evident from the firft Definition, that every ftraight line, excepting V C, drawn from V to the tangent C F will fall without the fuperficies V A C B. The plane paffing through V C, C F, therefore, can only meet the fuperficies in the ftraight line V C, and the curve of the fection G D H in the point D only. Confequently as D E is in the plane paffing through V C, C F, D E touches the fection G D H, according to the twentieth Definition.

But no other ftraight line can touch the fection G D H in the point D. For, if it be poffible, let D I touch the fection, and then as D I meets the curve G D H only in the point D, it can meet the fuperficies in that point only, and it will therefore touch the fuperficies in D. Moreover as no ftraight line, excepting D E the line of common fection, can be in the plane of the fection G D H and alfo in the plane V C F, and as D I, according to hypothefis and the twentieth Definition, is in the plane of the fection G D H, D I is not in the plane V C F. Let a plane be paffed through the ftraight lines V D C, D I, and let it cut the plane of the bafe in the ftraight line K C. Then as the ftraight line D I touches the fuperficies, every point in it, excepting D, falls without the fuperficies, according to the tenth Definition. It is therefore evident, from the firft Definition, that every ftraight line drawn from V, excepting V D C, in the plane paffing through V D C, D I, will

 fall

BOOK I.

fall without the ſuperficies. The plane paſſing through V D G, D I will therefore meet the ſuperficies in the ſtraight line V D C only; and conſequently K C will touch the baſe A C B. Again, as the planes V C K, V C F, cut one another in the ſtraight line V C, the ſtraight lines K C, C F are not in the ſame ſtraight line. The two ſtraight lines K C, C F therefore touch the circle A C B in the point C, which (16. iii.) is abſurd. Conſequently no other ſtraight line, beſides D E, can be drawn to touch the ſection G D H in the point D.

Cor. 1. If a ſtraight line as C F touch the baſe of the cone in the point C, and from V, the vertex, the ſide V C be drawn; a plane paſſing through C F, C V will touch the conical ſuperficies in the ſide V C; and it is evident from the firſt Definition (and 1. xi.) that this plane produced on the other ſide of V will touch the oppoſite ſuperficies in C V produced.

Cor. 2. If a ſtraight line as D E touch the conical ſuperficies, or a conic ſection G D H, and a ſide V D C of the cone be drawn through D the point of contact, a plane paſſing through this ſide of the cone and the tangent D E will touch the ſuperficies of the cone; and being produced beyond V, it will touch the oppoſite ſuperficies in V C produced. This is evident from the demonſtration of the Propoſition, and the preceding Cor.

Cor. 3. If the ſection G D H be an hyperbola, the tangent D E cannot meet the oppoſite hyperbola. For D E is the common interſection of the plane V C F and the plane of the ſection G D H, and, by the firſt Cor. the plane V C F touches the oppoſite ſuperficies in C V produced. It is therefore evident from the eighteenth Definition that the tangent D E cannot meet the oppoſite hyperbola.

PROP.

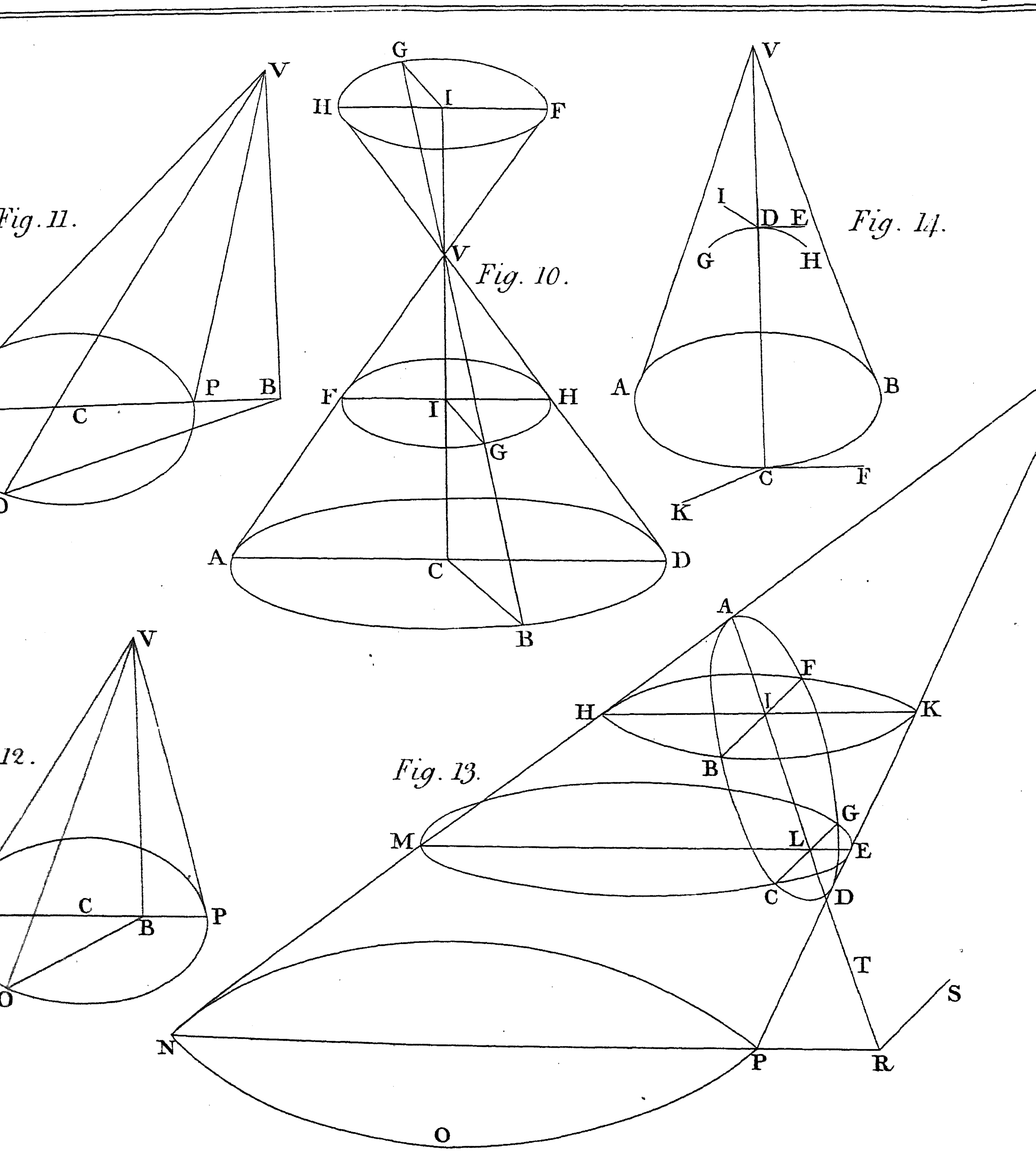


Fig. 10. Fig. 11. Fig. 12. Fig. 13. Fig. 14.

B4

PROP. VII.

BOOK I.

If a ſtraight line paſs through the vertex and fall without the oppoſite cones, two planes, and only two, can be drawn through it to touch the conical ſuperficies; and theſe planes will be on the oppoſite ſides of a plane paſſing through the ſtraight line, and cutting the baſe.

Let the ſtraight line V G paſs through V the vertex, and fall without the oppoſite cones V A M B, D V E; two planes, and only two, can be drawn through it to touch the ſuperficies; and if a plane paſs through V G, and cut the baſe in the ſtraight line C F, the tangent planes will be on the oppoſite ſides of the plane paſſing through V G, C F. Fig. 18. and 19.

Firſt let the ſtraight lines V G, C F be parallel. Let C F be biſected in L, and draw I L M at right angles to C F, and let it meet the circumference of the circle in the points M, I. Let a plane paſs through the ſtraight line V G and the point I, and this plane will touch the ſuperficies. For let it cut the plane of the baſe in the ſtraight line I K. Then as C F, V G are parallel, and as the plane of the baſe paſſes through C F, and the plane V I K paſſes through V G, the interſection I K of theſe planes will be parallel to C F, by the ſecond Lemma. But as I M biſects C F at right angles, it paſſes through the center of the circle (Cor. 1. iii.), and as I K, C F are parallel, and as the angle I L F is a right one, the angle K I L is alſo a right one (29. i.), and therefore I K (16. iii.) touches the circle in I. Conſequently, by Cor. 1. Prop. VI. the plane paſſing through V I, I K, or, as above, through V G, V I, touches the ſuperficies; and in the ſame manner it may be proved that the plane paſſing through V G and the point M touches the ſuperficies. Fig. 18.

Secondly, let the ſtraight line V G be not parallel to Fig. 19.

BOOK I. CF, but let it meet it in K. Draw KI, KM (17. iii.) touching the base in I, M; and then the plane passing through VG, IK, and also that passing through VG and MK, will touch the superficies, by Cor. 1. Prop. VI.

It is evident, in either case, that no other plane, besides the above-mentioned two, can pass through VG, and touch the superficies; and that one of these tangent planes is on the one side, and the other on the opposite side of the plane passing through VG, CF.

PROP. VIII.

If a conic section surround the cone, two straight lines, and only two, parallel to one another, can be drawn to touch the section; if the section does not surround the cone, no straight line, parallel to a tangent, can be drawn to touch the section; but if the section be an hyperbola, one straight line, and one only, parallel to a tangent, can be drawn to touch the opposite hyperbola.

Fig. 16. Part I. Let the section DHLK surround the cone VAFB, and let the straight line GD touch the section in the point D: another straight line, and only one, parallel to GD, can be drawn to touch the section.

For let VDA be the side of the cone passing through D, the point of contact, and let a plane pass through VA, DG, and this plane will touch the conical superficies in the side VA, by Cor. 2. Prop. vi. In this plane, and through V, the vertex of the cone, draw VT parallel to DG. Then VT will fall without the opposite cones; and by Prop. VII. another plane can be passed through VT touching the conical superficies. Let this plane touch the superficies in the side VLB, and let its intersection with the plane of the section DHLK be LI. Then as LI is in the plane touching the

the cone in the ſide V L B, it meets the conical ſuperficies in the point L only. It will therefore meet the curve of the ſection D H L K in the point L only, and conſequently it will touch the ſection: and as the plane T V L I paſſes through V T, and the plane of the ſection paſſes through D G parallel to V T, by the ſecond Lemma L I is parallel to D G. And as no other plane paſſing through V T can touch the conical ſuperficies, beſides the two T V A, T V B, it is evident, from the ſecond Lemma, that no other ſtraight line beſides L I, parallel to D G, can be drawn to touch the ſection D H L K. BOOK I.

Part II. Let F D C be a ſection which does not ſurround the cone, and let D G touch the ſection in the point D. No other ſtraight line, parallel to D G, can be drawn to touch the ſection. Fig. 15. and 17.

For let V B E be the vertical plane to the parabola, or hyperbola, as in the fifteenth, ſixteenth, eighteenth, and nineteenth Definitions. Let V D A be the ſide of the cone paſſing through D, the point of contact; and through V D A, D G let a plane paſs, and let it cut the vertical plane in the ſtraight line V T. Then, by Cor. 2. Prop. VI. the plane V D G will touch the conical ſuperficies, and (16. xi.) D G, V T will be parallel; and as V T is in the plane, touching the conical ſuperficies in the ſide V D A, it will fall without the oppoſite cones. Another plane, therefore, and only one, can be paſſed through V T to touch the conical ſuperficies, by Prop. VII. But when the ſection is a parabola, the other plane paſſing through V T and touching the ſuperficies is the vertical plane V B E, which is parallel to the parabola. When the ſection is an hyperbola, then the vertical plane V B E paſſes through V T, and cuts the baſe of the cone in the ſtraight line B E; and ſuppoſing T V L to be the other plane paſſing through V T, and

BOOK I.

and touching the conical fuperficies, the planes T V L, V D G are on oppofite fides of V B F, by the feventh Propofition. Confequently the plane T V L cannot meet the hyperbola F D C. It therefore follows from the above, and the fecond Lemma, that if the fection does not furround the cone, no ftraight line, parallel to a tangent, can be drawn to touch the fection.

Fig. 17.

Part III. Let F D C, Q R S be oppofite hyperbolas, and let D G touch the hyperbola F D C in the point D. Then one ftraight line, and only one, parallel to D G can be drawn to touch the oppofite hyperbola Q R S.

For, every thing remaining as in the preceding part, through the fides V A, V L, in which the planes T V A, T V L, paffing through T V parallel to D G, touch the fuperficies, let a plane be paffed, cutting the vertical plane in the ftraight line V W and the plane of the hyperbolas in the ftraight line D R. Then as V W, R D, L V are in the fame plane, and as (16. xi.) V W, R D are parallel, and L V meets V W, it will alfo meet R D, by the third Lemma. Let them meet in the point R. Then as the plane T V L touches the oppofite cone M V N in L V produced, it will meet the plane of the hyperbolas in the point R. Let the interfection of thefe two planes therefore be R X; and as the plane of the hyperbolas paffes through D G, and the plane T V L paffes through T V parallel to D G, by the fecond Lemma R X is parallel to D G; and being in the plane touching the conical fuperficies, it will touch the hyperbola Q R S in the point R. It is alfo evident, for the fame reafons as are mentioned above, that no other ftraight line parallel to R X, or D G, can be drawn to touch the hyperbola Q R S.

Cor. 1. If a ftraight line touch a conic fection, a ftraight line drawn through any point within the fection, and parallel to it, will meet the curve in two points.

points. For let every thing remain as in the demonſtration of the Propoſition, and let P be any point within the ſection. Through V T and the point P let a plane be paſſed, and let this plane cut the plane of the ſection in the ſtraight line H K; and, by the ſecond Lemma, H K will be parallel to G D the tangent, and alſo to V T. Now as the point P is within the ſection it is alſo within the cone, and therefore, by the firſt Propoſition, the plane paſſing through V T and the point P will cut the cone in two ſides; and as theſe two ſides and V T, H K are in the ſame plane, and V T, H K are parallel, H K will meet each of theſe two ſides, by the third Lemma; and as H K is in the plane of the ſection, it muſt meet the curve of the ſection in the ſame points in which it meets theſe two ſides of the cone. It is alſo evident from the Propoſition, and the preceding part of this Corollary, that a ſtraight line drawn through any point within the oppoſite hyperbola Q R S, and parallel to G D, will meet the curve Q R S in two points. Fig. 15, 16, and 17.

Cor. 2. If a ſtraight line meet the curve of a conic ſection in two points, two ſtraight lines may be drawn parallel to it to touch the ſection, if it ſurround the cone; but if the ſection does not ſurround the cone, only one ſtraight line parallel to a ſecant can be drawn to touch the ſection; and, if the ſection be an hyperbola, only one ſtraight line parallel to a ſecant can be drawn to touch the oppoſite hyperbola. For, let H, K be the points in which the ſecant H K meets the curve of the ſection; and through H K and V, the vertex of the cone, let a plane be paſſed, and, if the ſection ſurround the cone, draw V T in this plane parallel to H K. Then as this plane, by the Cor. to Prop. I. can only cut the oppoſite ſuperficies in ſtraight lines drawn through V the vertex and the points H, K, it is evident, Fig. 16.

BOOK I. dent that V T muſt fall without the oppoſite cones. Conſequently, by Prop. VII. two planes can be paſſed through V T to touch the conical ſuperficies, one on each ſide of H K; and the interſections of theſe planes with the plane of the ſection will touch the ſection, and, by the ſecond Lemma, theſe tangents will be parallel to H K. If the ſection does not ſurround the cone, let the plane paſſing through the ſecant H K and V cut the vertical plane V B E in the ſtraight line V T, and, for the ſame reaſons as are mentioned above, V T will fall without the oppoſite cones. Then through V T two planes may be paſſed, by Prop. VII. touching the conical ſuperficies. But, according to the demonſtration of the Propoſition, only one of theſe planes can meet the parabola, and one of them can meet the hyperbola, and the other the oppoſite hyperbola; and, by the ſecond Lemma, G D, the interſection of the plane T V A with the plane of the ſection F D C, will be parallel to H K, and R X the interſection of the plane T V L with the plane of the oppoſite hyperbolas, will alſo be parallel to H K, and each of the ſtraight lines G D, R X muſt touch the ſection which it meets.

Fig. 15. and 17.

PROP. IX.

If a ſtraight line meet the curve of a conic ſection in two points, any ſtraight line parallel to it, drawn through a point within the ſame ſection, or, if the ſection be an hyperbola, within the oppoſite hyperbola, will alſo meet the curve of the ſection, in which it is drawn, in two points. And if a ſtraight line meet each of the curves of two oppoſite hyperbolas in one point, a ſtraight line parallel to it, drawn through any point in the plane of theſe ſections, will alſo meet each of the curves of theſe oppoſite hyperbolas in one point.

Part

Fig. 15.

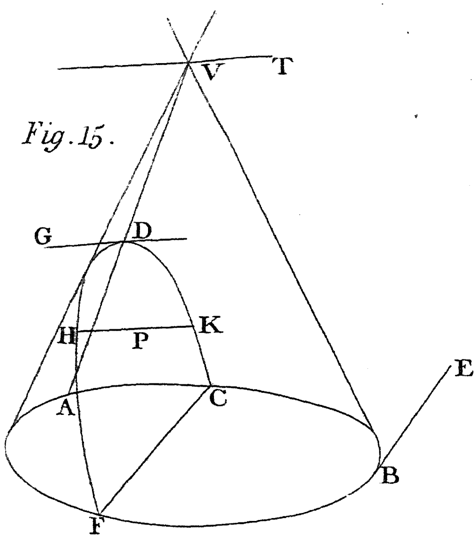


Fig. 17.

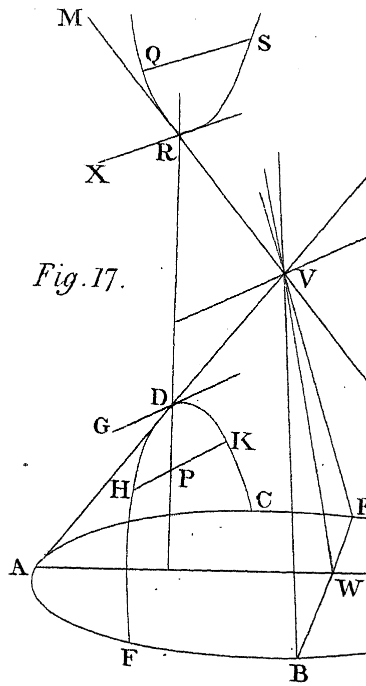


Fig. 16.

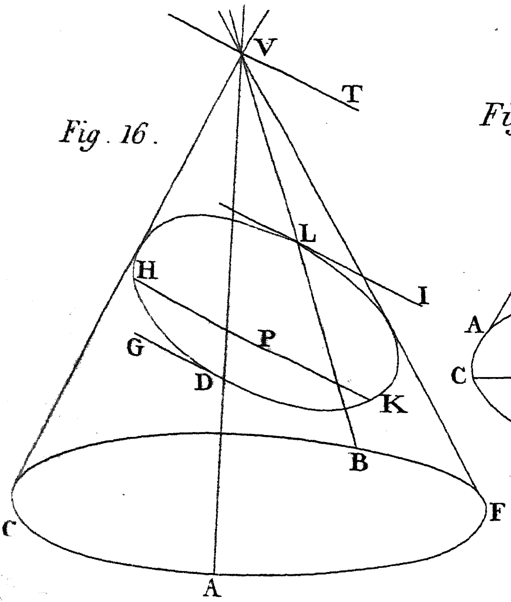
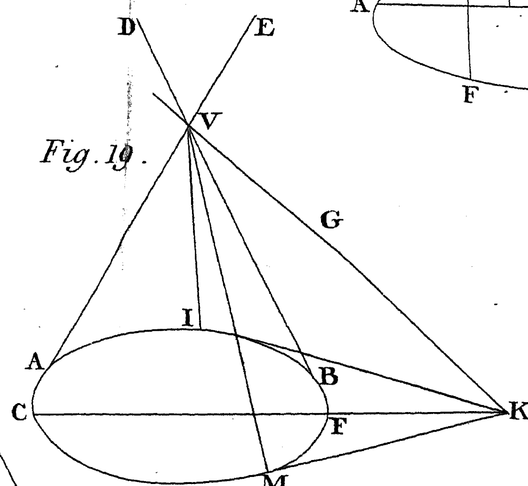
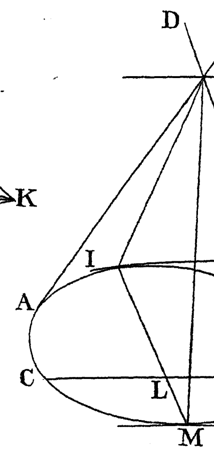


Fig. 19.





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B

Part I. Let A C D B be a conic ſection, and let the ſtraight line C D meet the curve in the points C, D; a ſtraight line, as A B, drawn parallel to C D through any point P within the ſection will meet the curve in two points. Fig. 20.

For let V be the vertex of the cone V E G H F, in which the ſection is formed, and through C D and V let a plane be paſſed, and let it cut a plane paſſing through A B and V in the ſtraight line V T. Then the plane paſſing through C D and V muſt cut the cone in the ſides V C, V D, and, by the ſecond Lemma, V T is parallel to C D and alſo to A B. Again, as V T is in the plane V C D, and as this plane, by Cor. Prop. I. cuts the oppoſite ſuperficies only in V C, V D, or in theſe lines produced, it follows that V T falls without the oppoſite cones. The plane paſſing through V T, A B will therefore cut the cone in two ſides, and, by the third Lemma, A B will meet each of theſe two ſides in the ſuperficies of the cone, and being in the plane of the ſection, it muſt meet the curve in the ſame points. If the ſection be an hyperbola, it is evident, for the ſame reaſons, that a ſtraight line drawn parallel to C D, through any point within the oppoſite hyperbola will meet the curve in two points.

Part II. Let A C, B D be two oppoſite hyperbolas, and let the ſtraight line C D meet the curve of one of them in the point C, and the curve of the other in the point D. Through the point P in the plane of theſe ſections let the ſtraight line A B be drawn parallel to C D; A B will meet each of the oppoſite hyperbolas A C, B D in one point. Fig. 21.

For let V be the vertex of the oppoſite cones, in which the hyperbolas are formed, and through C D and V let a plane be paſſed, and let it cut a plane paſſing through A B and V in the ſtraight line V T. Then the

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the plane paffing through C D and V muft cut the oppofite cones in the fides V C, V D; and, by the fecond Lemma, V T is parallel to C D and alfo to A B. Again, as V T is in the plane V C D, and as this plane, by Cor. Prop. I. cuts the oppofite fuperficies only in V C, V D, or in thefe lines produced, it follows that V T falls within the oppofite cones. The plane paffing through V T, A B will therefore cut the oppofite cones in two fides; and from the above, and the third Lemma, A B will meet one of thefe two fides in the one fuperficies, and the other in the oppofite fuperficies. But as A B is in the plane of the oppofite hyperbolas A C, B D, it muft meet the curve of one of the hyperbolas and the fuperficies, in which this hyperbola is, in the fame point. The ftraight line A B will therefore meet the curve of each of the oppofite hyperbolas in one point.

Cor. If a ftraight line as C D meet the curve of a conic fection in two points, it will fall wholly within the fection, but being produced it will fall without the fection. If a ftraight line meet each of the curves of oppofite hyperbolas in one point, it will fall wholly without the hyperbolas, but being produced it will fall on the one fide within one hyperbola, and on the other within the oppofite hyperbola. For it is evident from the demonftration of the Propofition that C D in Fig. 20. is within the cone, and that being produced it muft fall without it; and in Fig. 21. it is evident that C D is without the oppofite cones, and that being produced it falls on the one fide within one cone, and on the other fide within the oppofite cone.

PROP. X.

If a ftraight line cut either or both of the oppofite conical fuperficies, and meet a ftraight line which is parallel to the bafe of the cone, and which cuts either of the oppofite fuper-

*ſuperficies, the rectangle under the ſegments of the firſt mentioned line will be to the rectangle under the ſegments of the other in the ſame ratio, wherever the point of concourſe may be in the firſt mentioned line **.

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Let the ſtraight line F G H cut either or both of the oppoſite ſuperficies in the points G, H, and meet, in the point F, the ſtraight line F D E parallel to the baſe of the cone, and cutting either of the oppoſite ſuperficies in the points E, D; the rectangle under G F, F H will be to the rectangle under D F, F E in the ſame ratio, wherever the point of concourſe F may be in the ſtraight line F G H. Fig. 26, 27, 28.

Caſe 1. If the ſtraight line F G H be alſo parallel to the plane of the baſe, then the ſection G D H E, formed by the cone and the plane paſſing through F G H, F E D will be a circle, by the fourth Lemma, and Prop. II. and therefore (35 or Cor. 36. iii.) the rectangle under G F, F H will be to the rectangle under D F, F E in the ratio of equality. Fig. 26.

Caſe 2. Let F G H be not parallel to the baſe of the cone. Through F E D let a plane be paſſed parallel to the baſe, and let the ſection formed by it with the cone be the circle D E I K, as in the ſecond Propoſition. Through the points G, H, and V, the vertex of the cone, let a plane be paſſed, and let it cut the ſuperficies in the ſides A V I, B V K, the plane of the baſe in the ſtraight line A B L, and the plane of the circle D E I K in the ſtraight line I F K; and in the plane V G H draw V L parallel to G H, and let it meet A B L in the point L. Fig. 27, 28.

* When two ſecants meet one another, the ſegments of either of the two are its parts between the point of concourſe and the points in which it meets the ſuperficies; and if a tangent meet a ſecant, or another tangent, its magnitude is limited by the point of concourſe, and its point of contact.

Then

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Then as the ſtraight lines A B L, I F K (16. xi.) are parallel to one another, and the ſtraight lines V L, F G H alſo parallel to one another, in the triangles V L B, H F K, the angles (29. i.) B V L, V B L in the one are equal to the angles K H F, H K F in the other, each to each; and in the triangles V L A, G F I, the angles V A L, A V L in the one are equal to the angles G I F, I G F in the other, each to each. The triangles V L B, H F K are therefore equiangular to one another, as are alſo the triangles V L A, G F I to one another. Hence (4. vi.)

V L : L A : : G F : F I, and

V L : L B : : H F : F K, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GF \times FH : IF \times FK$. But (35 or Cor. 36. iii.) $IF \times FK = DF \times FE$, and conſequently,

$$VL^2 : AL \times LB :: GF \times FH : DF \times FE.$$

Cor. 1. If the ſtraight line F G H meet the ſtraight line F P parallel to the baſe, and touching either ſuperficies in the point P, the rectangle under G F, F H will be to the ſquare of F P in the ſame ratio, wherever the point of concourſe may be in the line F G H, as is evident from the above (and 36. iii.) And if the ſtraight line M V, paſſing through V the vertex, be parallel to the baſe, and meet G H in M, M V is to be conſidered as a tangent; for as above, by ſimilar triangles,

V L : L A : : G M : M V, and

V L : L B : : H M : M V, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GM \times MH : MV^2$.

Cor. 2. If a ſtraight line cut either or both the oppoſite ſuperficies, and meet a ſtraight line parallel to the baſe, and touching or cutting either ſuperficies; the rectangle under the ſegments of the firſt mentioned line will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in

in a conftant ratio, wherever the point of concourfe may be in the firft mentioned line. For this ratio will be either that of equality, as in the firft cafe of the demonftration, or it will be that of VL^2 to $AL \times LB$, as in the fecond cafe, and in the preceding Cor. BOOK I.

PROP. XI.

If a ftraight line touch either of the oppofite fuperficies, and meet a ftraight line parallel to the bafe of the cone, and which cuts either of the oppofite fuperficies, the fquare of the tangent will be to the rectangle under the fegments of the fecant in the fame ratio, wherever the point of concourfe may be in the tangent.

Let the ftraight line TF touch either of the oppofite fuperficies in the point T, and meet in the point F the ftraight line FE, parallel to the bafe of the cone, and cutting either of the oppofite fuperficies in the points G, E; the fquare of the tangent TF will be to the rectangle under GF, FE, in the fame ratio, wherever the point F may be in the tangent TF. Fig. 22. 23.

Cafe 1. If the tangent TF be alfo parallel to the plane of the bafe, then the fection TGE, formed by the cone and the plane paffing through TF, FE will be a circle, by the fourth Lemma, and Prop. II, and the fquare of TF (36. iii.) will be to the rectangle GF, FE in the ratio of equality. Fig. 22.

Cafe 2. Let TF be not parallel to the bafe of the cone. Through FE let a plane be paffed parallel to the bafe, and let the fection formed by it with the cone be the circle DGE, as in the fecond Propofition. Through the tangent TF and V, the vertex of the cone, let a plane be paffed, and let it touch the fuperficies in the fide VTD, according to Cor. 2. Prop. VI. and let it cut the plane of the bafe in the ftraight line CL, and Fig. 23.

BOOK I.

the plane of the circle D G E in the ſtraight line D F. In the plane V C L draw V L parallel to T F, and let it meet the baſe in the point L. Then as the ſtraight lines C L, D F (16. xi.) are parallel to one another, and the ſtraight lines V L, T F alſo parallel to one another, in the triangles V C L, T D F, the angles (29. i.) C V L, V C L in the one are equal to the angles D T F, T D F in the other, each to each.

The triangles V C L, T D F are therefore equiangular, and conſequently (4. vi.)

$$VL : LC :: TF : FD,$$ and by the fifth Lemma,

$$VL^2 : LC^2 :: TF^2 : FD^2.$$

But (36. iii.) FD^2 is equal to the rectangle, under G F, F E, and conſequently $VL^2 : LC^2 :: TF^2 : GF \times FE$.

Cor. 1. If a ſtraight line, as T F, touch either of the oppoſite ſuperficies in T, and meet a ſtraight line, as F D, parallel to the baſe of the cone, and which touches either ſuperficies as in D; the ſquare of T F will be to the ſquare of F D in the ſame ratio, wherever the point of concourſe F may be in T F. If M V, paſſing through V the vertex, be parallel to the baſe and meet T F in M, then M V is to be conſidered as a tangent; and as above $VL^2 : LC^2 :: TM^2 : MV^2$.

Cor. 2. If a ſtraight line touch either of the oppoſite ſuperficies, and meet a ſtraight line parallel to the baſe, and touching or cutting either ſuperficies; the ſquare of the firſt mentioned tangent will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a conſtant ratio, wherever the point of concourſe may be in the firſt mentioned tangent. For this ratio will be that of equality, as in the firſt caſe of the demonſtration, or it will be that of VL^2 to LC^2, as in the ſecond caſe.

PROP.

PROP. XII.

If the firſt of two ſtraight lines be parallel to the ſecond and touch or cut either or cut both of the oppoſite ſuperficies, and if the ſecond alſo touch or cut either, or cut both of the oppoſite ſuperficies, and if each of the two meet a ſtraight line parallel to the baſe of the cone and touching or cutting either ſuperficies; then the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent or the rectangle under the ſegments of the ſecant which it meets, as the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets.

Caſe 1. If the firſt and ſecond ſtraight lines, parallel to one another, be alſo parallel to the plane of the baſe, then the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a ratio of equality, as in the firſt caſe of the demonſtration of Prop. X. and XI. And, for the ſame reaſons, the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant which it meets in a ratio of equality. In this caſe therefore the Propoſition is evident.

Caſe 2. Let the firſt and ſecond ſtraight lines, parallel to one another, be not parallel to the plane of the baſe. Suppoſe a plane to paſs through each of the two parallel lines, and v the vertex, and then theſe planes will cut one another, and their line of common ſection will paſs through v, and, by the ſecond Lemma, it will be parallel to each of the two ſtraight lines parallel to one

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one another. Let V L be their line of common ſection, as in Fig. 27. 28. and 23. and let it meet the baſe in the point L. Then the planes paſſing through the two parallel lines and V L, will cut the plane of the baſe in ſtraight lines paſſing through L; and each of theſe lines of common ſection with the plane of the baſe will cut or touch the baſe, as in the ſecond caſe of the demonſtration of the tenth and eleventh Propoſitions; and the rectangle under the ſegments of either, if a ſecant, or its ſquare, if a tangent, will be (35 and 36. iii.) equal to the rectangle under A L, L B, L A being a ſtraight line cutting the baſe in the points B, A. Conſequently by Cor. 2. Prop. X. and Cor. 2. Prop. XI. the ſquare of the firſt ſtraight line, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant, which it meets as the ſquare of V L to the rectangle under A L, L B. For the ſame reaſon, the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of the tangent, or the rectangle under the ſegments of the ſecant, which it meets as the ſquare of V L to the rectangle under A L, L B. Hence (11. v.) if the firſt of two ſtraight lines be parallel, &c.

SCHOLIUM.

As every point in the curve of a Conic Section is alſo in the conical ſuperficies, it is evident that all the Propoſitions demonſtrated concerning ſtraight lines touching or cutting the conical ſuperficies, or oppoſite ſuperficies, may be transferred to ſtraight lines, which in the ſame manner touch or cut a conic ſection, or oppoſite hyperbolas.

PROP.

PROP. XIII.

BOOK I.

Fig. 24.

If there be four ſtraight lines in the plane of a conic ſection, and if A B *the firſt meet* C B *the ſecond, and* D E *the third meet* F E *the fourth, and if the firſt be parallel to the third, and the ſecond to the fourth, and if each of them either touch or cut a conic ſection, or cut oppoſite hyperbolas; then the ſquare of* A B, *if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of* C B, *if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of* D E, *if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of* F E, *if a tangent, or the rectangle under its ſegments, if a ſecant.*

Caſe 1. If the ſtraight lines A B, C B, and conſequently D E, F E, be each parallel to the baſe of the cone in which the ſection was formed, or the baſe of the oppoſite cone, the ſection muſt be a circle, as in the firſt caſe of Prop. X. or Prop. XI, and the ratio above ſtated will be that of equality.

Caſe 2. Let A B, D E be not parallel to the baſe of the cone, in which, or in which and its oppoſite, the ſection or oppoſite hyperbolas were formed; but let C B, F E be parallel to the baſe, and then the Propoſition is evident from the twelfth Propoſition.

Caſe 3. Let neither A B nor C B, and conſequently neither D E nor F E, be parallel to the baſe of the cone; but ſuppoſe B G, E H to be ſtraight lines parallel to the baſe of the cone in which the ſection, or in which and the oppoſite cone the oppoſite hyperbolas were formed; and let B G, E H touch or cut either of the oppoſite conical ſuperficies. Then by the twelfth Propoſition, the ſquare of A B, if a tangent, or the rectangle under its ſegments, if a ſecant, will be to the ſquare of B G, if a tangent, or the rectangle under its ſegments, if a ſe-

cant,

BOOK I. cant, as the ſquare of D E, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of E H, if a tangent, or the rectangle under its ſegments, if a ſecant. Again, by the twelfth Propoſition and inverſion, the ſquare of B G, if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C B, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of E H, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of F E, if a tangent, or the rectangle under its ſegments, if a ſecant. Conſequently,

$$\left.\begin{matrix} t.\ AB^2 \\ \text{or} \\ ſ.\ AB^r \end{matrix}\right\} : \left\{\begin{matrix} t.\ BG^2 \\ \text{or} \\ ſ.\ BG^r \end{matrix}\right\} : \left\{\begin{matrix} t.\ CB^2 \\ \text{or} \\ ſ.\ CB^r \end{matrix}\right.$$

$$\left.\begin{matrix} t.\ DE^2 \\ \text{or} \\ ſ.\ DE^r \end{matrix}\right\} : \left\{\begin{matrix} t.\ EH^2 \\ \text{or} \\ ſ.\ EH^r \end{matrix}\right\} : \left\{\begin{matrix} t.\ FE^2 \\ \text{or} \\ ſ.\ FE^r \end{matrix}\right.$$

The ſquare of A B therefore (22. v.) if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C B, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of D E, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of F E, if a tangent, or the rectangle under its ſegments, if a ſecant.

Cor. If A B, C B, D E, F E be tangents, then it is evident (22. vi.) that A B : C B : : D E : F E.

PROP. XIV.

Any ſtraight line parallel to the ſide of a cone, provided it be not in the plane touching the cone in that ſide, will meet one of the oppoſite ſuperficies in one point, and in one point only.

Fig. 29. 30. Let the ſtraight line D C be parallel to V B a ſide of the cone V A M B, but not ſituated in the plane touching

ing the cone in the fide V B; the ftraight line D C will meet one of the oppofite fuperficies in one point, and in one point only. BOOK I.

Let a plane pafs through the parallels D C, V B, and as by hypothefis D C is not in the plane touching the cone in the fide V B, the plane paffing through D C, V B muft cut the cone. Let it cut the oppofite fuperficies therefore in the ftraight lines A V, B V. Then as A V meets B V in V the vertex, and as it is in the fame plane with the parallels V B, D C, by the third Lemma A V, or A V produced, muft alfo meet D C. Let them meet in D. Then D C muft meet one of the fuperficies in D, and as it is parallel to V B, it is evident it cannot meet the other fuperficies. It is alfo evident, that it can meet one of the oppofite fuperficies in one point only; for on one fide of D it is entirely within the fuperficies, and on the other entirely without it.

PROP. XV.

If a ftraight line parallel to a fide of the cone cut either of the oppofite conical fuperficies, and meet two ftraight lines parallel to the bafe of the cone, and which cut either fuperficies; the fegments of the firft mentioned line, between the fuperficies and the points of concourfe, will be to one another as the rectangles under the fegments of the fecants which it meets.

Let the ftraight line D C, parallel to V B a fide of the cone V A M B, cut either of the oppofite fuperficies in the point D, and meet in the points E, R the ftraight lines E I F, L R T, which are parallel to the bafe of the cone, and cut either fuperficies in the points I, F, and L, T; then the fegment D E is to the fegment D R as the rectangle under E I, E F to the rectangle under L R, R T. Fig. 29. 30.

 For

BOOK I.

For through the parallels D C, V B let a plane paſs, and let it cut the plane of the baſe in the ſtraight line A C B, and the ſuperficies in A V, B V. Through each of the ſtraight lines L R T, E I F let a plane paſs parallel to the baſe A M B, and let O L N T, I F H G be the circles formed, as in the ſecond Propoſition. Let E G H, O R N be the interſections of theſe circles and the plane paſſing through A V, B V. Let E G H meet A V in G and B V in H; and let O R N meet A V in O and B V in N. Then (16. xi.) the ſtraight lines E G H, O R N, A C B are parallel, and therefore (34. i.) E H, C B, R N are equal to one another; the angle D G E (29. i.) is equal to the angle D A C, and the angle D E G to the angle D C A. Hence (4. vi.) D E : E G : : D C : A C, and D R : O R : : D C : A C; and therefore (11. v.) D E : E G : : D R : O R. and (16. v.) D E : D R : : E G : O R. Conſequently (1. vi.) D E : D R : : E G × E H : O R × R N.

But (35. and 36. iii.) E G × E H = E I × E F, and O R × R N = L R × R T; and therefore D E : D R : : E I × E F : L R × R T.

Cor. If a ſtraight line parallel to a ſide of the cone cut either of the oppoſite conical ſuperficies, and meet two ſtraight lines parallel to the baſe, and which meet either ſuperficies; its ſegment between the ſuperficies, and the firſt of the two parallel to the baſe, will be to its ſegment between the ſuperficies and the ſecond, as the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant. For every thing remaining as above, if E P be parallel to the baſe, and touch either ſuperficies in P, E P will be in the plane of the circle I F H G and (36. iii.) the ſquare of E P will be equal to the rectangle under E I, E F. And, for the ſame reaſons, if the point

point R were without the circle, the ſquare of a ſtraight line parallel to the baſe, drawn from R and touching either ſuperficies would be equal to the rectangle under L R, R T. If a ſtraight line V K parallel to the baſe paſs through V, the vertex, and meet C D in K, V K is to be conſidered as a tangent. For D K : K V : : D C : C A, and D R : O R : : D C : C A. Hence (11. v.) D K : K V : : D R : O R, and (16. v.) D K : D R : : K V : O R. But K V, R N being parallel to the baſe are parallel to one another, and therefore (34. i.) K V, R N are equal, and conſequently (1. vi.) D K : D R : : KV^2 : O R × R N.

PROP. XVI.

If a ſtraight line cutting a parabola or hyperbola be parallel to a ſide of the cone in which the ſection is formed, and if it meet two ſtraight lines which are parallel to one another, and meet the ſame ſection, or the oppoſite hyperbolas; its ſegment between the curve and the firſt of the two parallels will be to its ſegment between the curve and the ſecond, as the ſquare of the firſt, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of the ſecond, if a tangent, or the rectangle under its ſegments, if a ſecant.

Suppoſe the ſtraight line B C to cut the curve of a parabola or hyperbola in the point A, and to be parallel to a ſide of the cone in which the ſection is formed, and let it meet the ſtraight lines B D, C E which are parallel to one another, and meet the ſame ſection, or the oppoſite hyperbolas; then A B is to A C as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant. Fig. 25.

For, as B C cuts the curve of the parabola or hyperbola

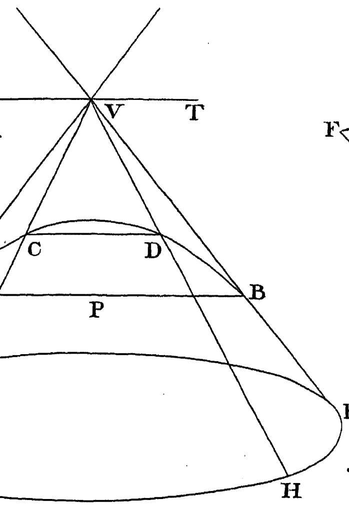
BOOK I.

bola in the point A, it will cut the conical ſuperficies in which the curve is formed in the ſame point; and the ſtraight lines B D, C E will meet the conical ſuperficies in the ſame points in which they meet the curve of the ſection, or the curves of the oppoſite hyperbolas. If therefore B D, C E be parallel to the baſe of the cone, the Propoſition is evident from the Cor. to Prop. XV. but if they are not parallel to the baſe of the cone, let B F, C G be parallel to the baſe of the cone, and let them meet the ſame or the oppoſite conical ſuperficies. Then by the Cor. to Prop. XV. A B is to A C as the ſquare of B F, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C G, if a tangent, or the rectangle under its ſegments, if a ſecant. Again, by Prop. XII. (and 16. v.) the ſquare of B F, if a tangent, or the rectangle under its ſegments, if a ſecant, is to the ſquare of C G, if a tangent, or the rectangle under its ſegments, if a ſecant, as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant. Conſequently (11. v.) A B is to A C as the ſquare of B D, if a tangent, or the rectangle under its ſegments, if a ſecant, to the ſquare of C E, if a tangent, or the rectangle under its ſegments, if a ſecant.

PROP. XVII.

If two ſtraight lines meeting one another touch a conic ſection, or oppoſite hyperbolas, and if a ſecant parallel to one of them meet the other and the ſtraight line joining the points of contact, the rectangle under the ſegments of the ſecant between the curve and the tangent will be equal to the ſquare of its ſegment between the tangent and the line joining the points of contact.

Let



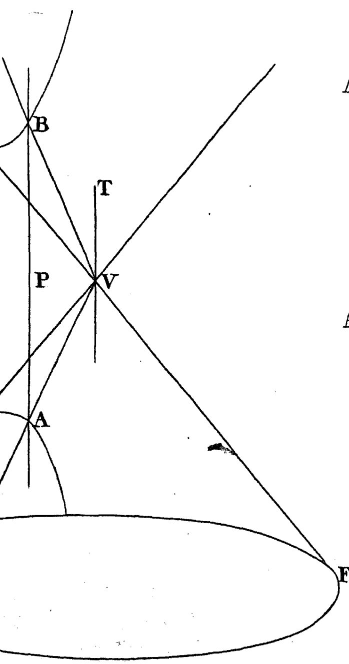


Fig. 23.

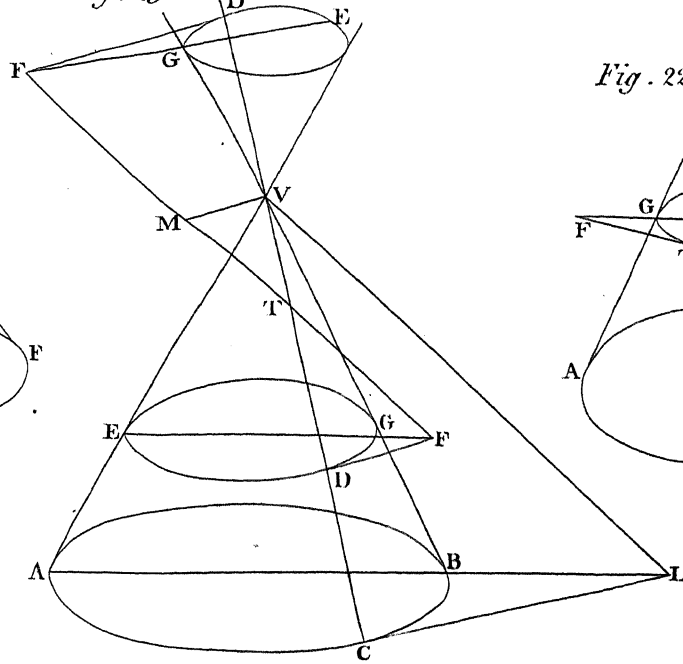


Fig. 22.

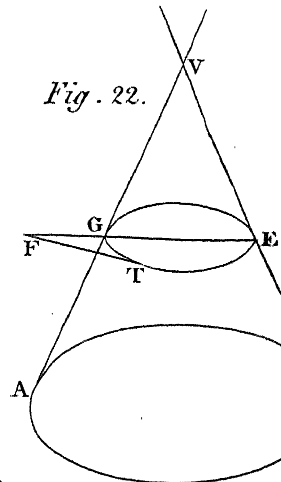


Fig. 24.

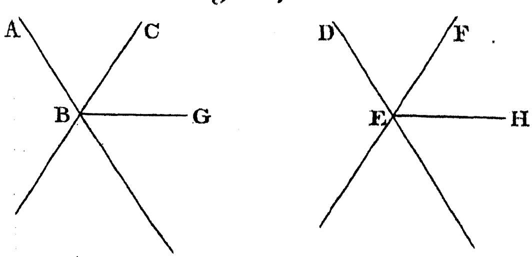
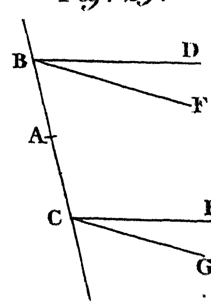


Fig. 25.



in the points A, C, and let the ſtraight line D P, parallel Fig. 31, 32.
to C F, cut the ſection or either of the oppoſite hyperbolas in D, P, and meet the tangent A F in G, and the ſtraight line A C, joining the points of contact, in B; then the rectangle under D G, G P is equal to the ſquare of G B. 33, 34.

For by Prop. XIII. $DG \times GP : AG^2 :: CF^2 : AF^2$. But as C F, B G are parallel, by Lemma V. (and 4. vi.) $CF^2 : AF^2 :: GB^2 : AG^2$, and therefore (11. v.) $DG \times GP : AG^2 :: GB^2 : AG^2$. Conſequently (14. v.) $DG \times GP = GB^2$.

Cor. 1. The reſt remaining as above, if from any point E in the tangent A F, there be drawn the ſtraight line E H parallel to the tangent C F, and meeting A C in H, and if from the ſame point E there be drawn any ſtraight line E I L, cutting the ſection or oppoſite hyperbolas in I and L; then the rectangle under I E, E L and the ſquare of E H will be to one another as the ſquares of the tangents, or the rectangles under the ſegments of the ſecants meeting one another and parallel to I L, E H. For from the point G draw G N parallel to I L, and let it cut the curve or curves in M, and N. Then by Prop. XIII. $IE \times EL : MG \times GN :: AE^2 : AG^2$; and by ſimilar triangles, and this Propoſition, $AE^2 : AG^2 :: EH^2 : GB^2$ or its equal $DG \times GP$. Hence (11. v.) $IE \times EL : MG \times GN :: EH^2 : DG \times GP$, and, by alternation, $IE \times EL : EH^2 :: MG \times GN : DG \times GP$. Conſequently, by Prop. XIII. (and 11. v.) the Cor. is evident.

Cor. 2. If a ſtraight line B H cut a conic ſection, or Fig. 35.
oppoſite hyperbolas in D, G, and meet in the points B, 36.
H two ſtraight lines A B, C H which touch the ſection or oppoſite hyperbolas in A, and C, and if B H meet

A C

BOOK I.

AC joining the points of contact in E; the rectangle under DB, BG will be to the rectangle under GH, HD as the square of BE to the square of HE. For if the tangents AB, CH be parallel, the triangles (29. i.) ABE, CHE will be equiangular, and therefore, by the fifth Lemma, (and 4. vi.) in this case $AB^2 : CH^2 :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : CH^2 :: DB \times BG : GH \times HD$. In this case therefore (11. v.) the Cor. is evident. But if the tangents be not parallel, through H draw the straight line LK parallel to AB, and let it meet AC in K, and the curve or curves in the points L, M. Then the triangles ABE, KHE are (29. i.) equiangular, and as above $AB^2 : HK^2 :: BE^2 : HE^2$. But by this Proposition HK^2 is equal to $LH \times HM$, and therefore $AB^2 : LH \times HM :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : LH \times HM :: DB \times BG : GH \times HD$. Consequently (11. v.) $DB \times BG : GH \times HD :: BE^2 : HE^2$.

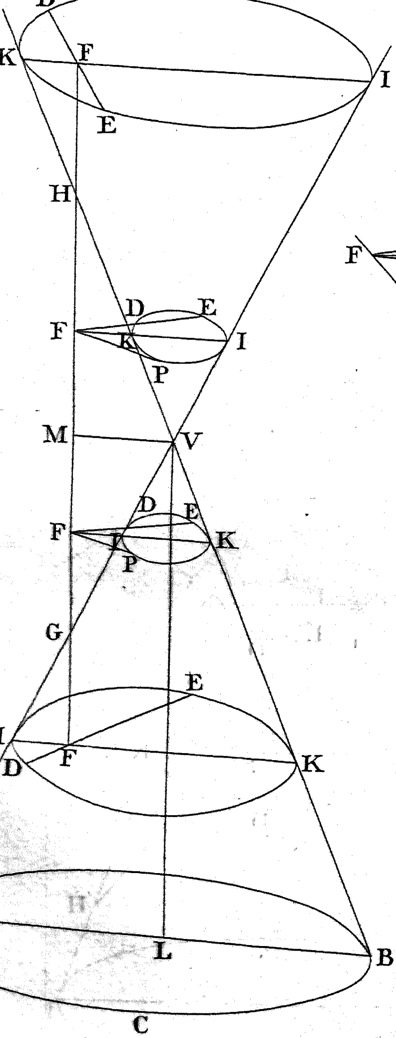

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Fig. 27.

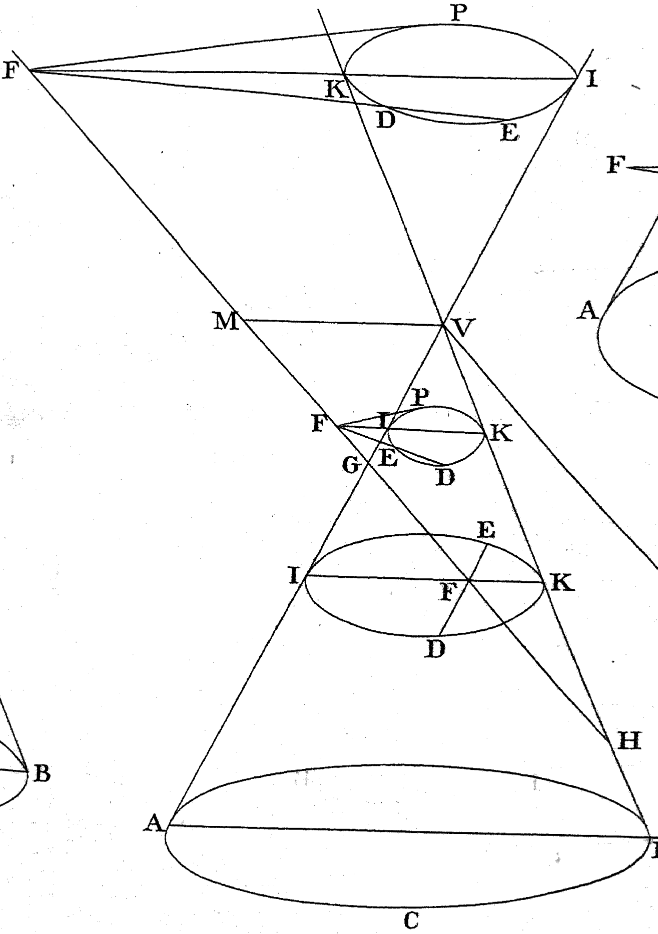
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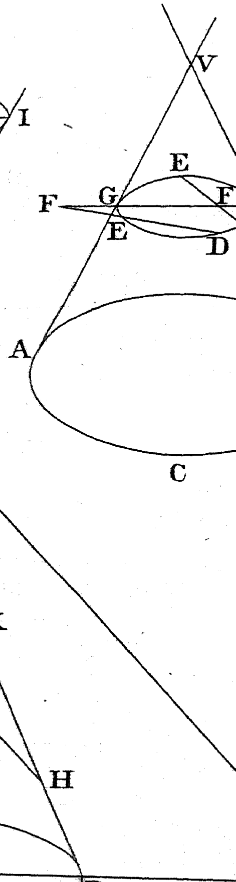
Fig. 26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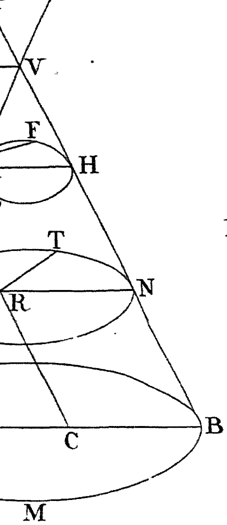


Fig. 31.

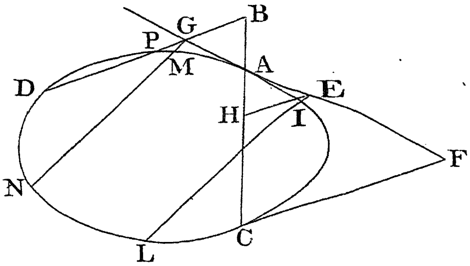


Fig. 30.

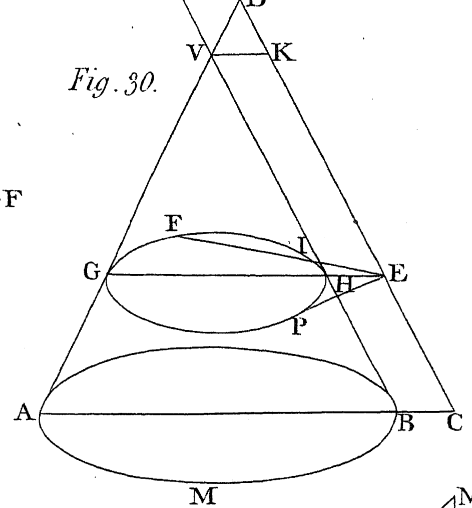


Fig. 32.

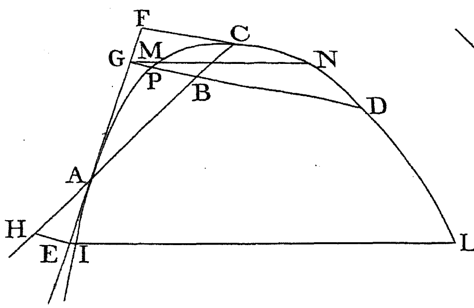


Fig. 33.

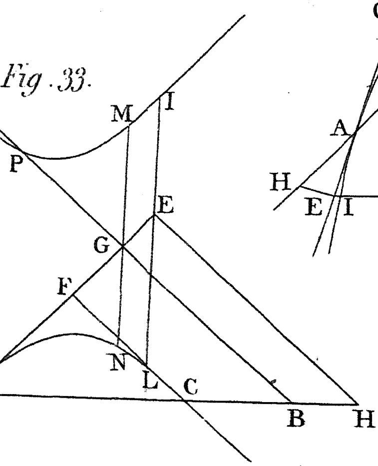
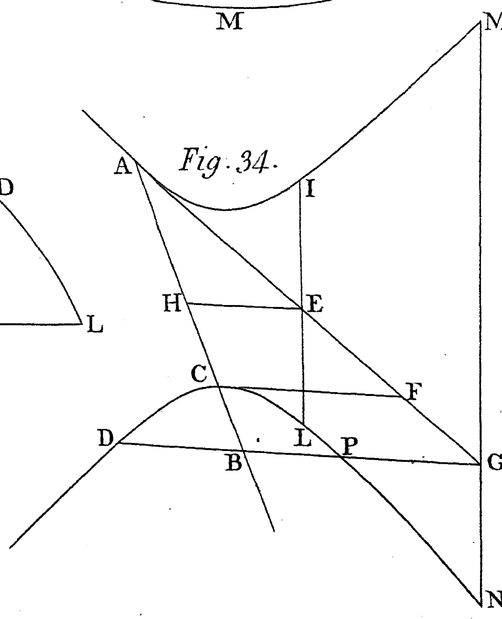


Fig. 34.



J. Basire s

A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

BOOK II.

Of the Ellipse and Hyperbola.

DEFINITIONS.

I.

THAT point within an ellipse or between opposite hyperbolas, in which every straight line, passing through it and terminated by the curve or opposite curves, is bisected, is called *the Center* of the ellipse, or *the Center* of the hyperbola or opposite hyperbolas.

II.

Any straight line passing through the center of an ellipse, and terminated by the curve, is called *a Diameter* of the ellipse.

III.

A straight line passing through the center of opposite hyperbolas, and terminated by the opposite curves, is called

BOOK II.

called *a Tranſverſe Diameter* of the oppoſite hyperbolas, or of either of the oppoſite hyperbolas. And a ſtraight line paſſing through the center of oppoſite hyperbolas, and biſecting a ſtraight line not paſſing through the center, and terminated by the oppoſite curves, is called *a Second Diameter* of the oppoſite hyperbolas, or of either of the oppoſite hyperbolas.

IV.

Any ſtraight line not paſſing through the center of an ellipſe, or oppoſite hyperbolas, terminated by the curve of the ellipſe or either hyperbola, or by the oppoſite curves, and biſected by a diameter, is called *a Double Ordinate* to the biſecting diameter; and its half is ſimply called *an Ordinate* to it.

V.

The points in which any diameter of an ellipſe meets the curve, or in which any tranſverſe diameter of oppoſite hyperbolas meets the oppoſite curves, are called *the Vertices* of the diameter; and the ſegments of a diameter, between an ordinate and its vertices, are called *Abſciſſes*.

VI.

Two diameters of an ellipſe, or oppoſite hyperbolas, of which each biſects all ſtraight lines terminated by the curve, or oppoſite curves, and parallel to the other, are called *Conjugate Diameters*.

VII.

A diameter of an ellipſe, or oppoſite hyperbolas, which cuts its ordinates at right angles is called *an Axis* of the ellipſe, hyperbola, or oppoſite hyperbolas.

PROP. I.

If two parallel ſtraight lines touch an ellipſe, or oppoſite hyperbolas, the ſtraight line joining the points of contact will

will bisect any straight line parallel to them, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas. BOOK II.

Let the two parallel straight lines G H, L M touch the ellipse A D B E, or the opposite hyperbolas F B N, I A K, in the points B, A; the straight line A B, joining the points of contact, will bisect any straight line F N, parallel to the tangents, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas. Fig. 37. 38.

Let F N meet the curve in the points F, N, and A B in O. Through the points F, N draw G L, H M parallel to A B. Let G L meet the curve of the ellipse again, or the curve of the opposite hyberbola in I, the tangent G H in G, and the tangent L M in L. Let H M meet the curve of the ellipse again, or the curve of the opposite hyperbola in K, the tangent G H in H, and the tangent L M in M. Then, by Prop. XIII. Book I. $BG^2 : IG \times GF :: AL^2 : FL \times LI$; and (34. i.) as B G is equal to A L, and therefore the square of B G equal to the square of A L, we have (14. v.) $IG \times GF$ equal to $FL \times LI$. Consequently, by the sixth Lemma, F G is equal to I L. For the same reasons N H is equal to K M, and (34. i.) as F G is equal to N H, I L is equal to K M; and (34. i.) as L G is equal to M H, I G is equal to K H, and $IG \times GF$ is equal to $KH \times HN$. Again, by Prop. XIII. Book I. $IG \times GF : GB^2 :: KH \times HN : HB^2$; and therefore (14. v.) the square of G B is equal to the square of H B, and G B is equal to H B. Consequently (34. i.) F O is equal to O N.

If, in the ellipse, the straight line D E, meeting the curve in D, E, be parallel to the tangents G H, L M, and if the straight line P D T parallel to A B touch the ellipse in D, then the straight line R E Q parallel to A B will

BOOK II.

will touch the ellipſe in E, and D E will be biſected in C, the point in which it meets A B. For let P D T meet the tangent G H in T, and the tangent L M in P; and let R E Q meet the tangent G H in Q, and the tangent L M in R. Then, by Prop. XIII. Book I. $BT^2 : DT^2 :: AP^2 : PD^2$; and (34. i.) as B T, A P are equal; the ſquare of B T is therefore equal to the ſquare of A P, and conſequently (14. v.) the ſquare of D T is equal to the ſquare of D P, and D T is equal to D P. But (34. i.) E Q, D T are equal to one another, as are alſo E R, D P to one another; and therefore E Q, E R are equal to one another. Now if R E Q could meet the curve in any other point beſides E, as in V, then as B Q (34. i.) is equal to A R, it might be proved as above, by means of Prop. XIII. Book I. that V Q × Q E is equal to E R × R V. It would therefore follow, by the ſixth Lemma, that E Q is equal to R V; which by the above is abſurd. Conſequently R E Q touches the ellipſe in E, and therefore, by the Cor. to Prop. XIII. Book I. E Q : Q B :: D T : T B; and as E Q, D T are equal, we have (14. v.) B Q equal to B T. Conſequently C D is equal to C E, for (34. i.) C E is equal to B Q, and C D to B T.

Cor. If two parallel ſtraight lines, as G H, L M touching an ellipſe or oppoſite hyperbolas meet a ſtraight line, as F I, which cuts the ellipſe or oppoſite hyperbolas, and is parallel to the ſtraight line joining the points of contact, the ſegments of the ſecant between the curve or curves and the tangents will be equal to one another; for by the above F G is equal to I L. And if two parallel ſtraight lines touching an ellipſe meet a ſtraight line which touches the ellipſe, and is parallel to the ſtraight line joining the points of contact, the ſegments of the laſt mentioned tangent, between the point of contact and the parallels, will be equal to one another.

:her. This is alſo evident from the above, for it proved that D P is equal to D T. BOOK II.

PROP. II.

wo parallel ſtraight lines touch an ellipſe or oppoſite yperbolas, and the ſtraight line joining the points of conzct be biſected, the point in which it is biſected will be e center of the ellipſe or oppoſite hyperbolas; and no her point can be a center of the ellipſe or oppoſite hyperbolas.

.et the two parallel ſtraight lines G H, L M touch ellipſe A D B E, or oppoſite hyperbolas F B N, I A K he points B, A, and let the ſtraight line A B, joining points of contact, be biſected in C; the point C is center of the ellipſe or oppoſite hyperbolas, and no er point, beſides C, can be the center of the ellipſe ppoſite hyperbolas. Fig. 37. 38.

'art I. Take any other point N in the curve of the pſe, or in the curve of either of the oppoſite hyperas. Then N C being drawn, and produced, it will et the curve of the ellipſe, or the curve of the oppohyperbola, and the whole line terminated by the ve, or curves, will be biſected in C. For let N C be parallel to G H, L M, and draw N O F parallel to m, and let it meet A B in O, and the curve again in Make A S equal to B O, and through S draw I S K allel to the tangents G H, L M, or to N O F, and let e one of the points in which it meets the curve. en, by Prop. I. N F will be biſected in O, and S K l be half the whole line, of which it is a part, terated by the curve; and therefore, by Prop. XIII. ok I. $AO \times OB : NO^2 :: BS \times SA : KS^2$. But as , B O are equal to one another, $AO \times OB$ is equal to $\times SA$, and therefore (14. v.) the ſquare of N O is

E equal

BOOK II.

equal to the ſquare of K S, and N O is equal to K S. Let N C produced meet K S I in I, and then as C S, C O are equal, and (29. i.) the triangles N C O, I C S equiangular, it follows (26. i.) that N C is equal to C I, and I S is equal to N O; and therefore, by the above, I S is equal to S K. Conſequently, by Prop. I. the point I is in the curve, and therefore N C being produced, it meets the curve of the ellipſe again, or the curve of the oppoſite hyperbola, and the whole line N I, terminated by the curve, or curves, is biſected in C. The point C is therefore the center of the ellipſe or oppoſite hyperbolas; for in the ellipſe the ſtraight line parallel to the tangents G H, L M, and paſſing through C is alſo biſected in C, by Prop. I.

Part II. No other point, beſides C, can be the center of the ellipſe, or oppoſite hyperbolas. In the ellipſe this is evident; for if there could be another, then a ſtraight line paſſing through C and that other center, and terminated by the curve, would, by the ſecond Definition, be biſected in two points: which is abſurd.

Fig. 38.

Nor can the hyperbolas F B N, I A K have any other center beſides C. For through C draw D E parallel to G H, L M; and ſuppoſe the point D in this ſtraight line to be another center. Through D draw the ſtraight line F I parallel to A B, and let it meet the tangents in G, L, and the oppoſite curves in F, I. Then, by Cor. Prop. I. F G is equal to I L; and (34. i.) as G D is equal to B C, and D L to C A, and B C equal to C A, it follows that F D, D I are equal. But through D draw a ſtraight line Q T, not parallel to A B, and let it meet the curves in Q, T; which it evidently may do, by Prop. IX. Book I. as it may be drawn parallel to a ſtraight line drawn from a point in the curve F B N through C and meeting the oppoſite curve, by Part I. Let Q T meet the ſtraight lines F N, I K parallel to the tangents G H,

L M,

L M, in R and P. Let R be between the points F, N, and then P will be without I, K, or without the hyperbola I A K. Then, by the above, as F D is equal to D I, and (29. i.) the triangles F D R, I D P equiangular, it follows (26. i.) that R D is equal to P D. Consequently T Q is not bisected in D, and therefore D is not a center. Nor can any point out of the line D E be a center; for if it could, then a straight line drawn through it, and parallel to A B, and meeting the curves, would, by the above, be bisected by D E, and, according to the first Definition, it would also be bisected in this other center. The same straight line would therefore be bisected in two points: which is absurd. Consequently no other point besides C can be a center.

Cor. 1. A straight line, as A B, joining the points of contact of two parallel tangents G H, L M, is a diameter of the ellipse, or opposite hyperbolas; and D E drawn through the center C, and parallel to the tangents, or to the secants F N, I K parallel to them, is also a diameter. For in the ellipse D E is a diameter, according to the second Definition; and in the hyperbola D E is a second diameter, by the third Definition, as it bisects, by the above, any straight line parallel to A B, and terminated by the curves of the opposite hyperbolas. Fig. 37. 38.

Cor. 2. If two straight lines A L, B G touch an ellipse, or opposite hyperbolas in A, B, the vertices of the diameter A B, they will be parallel. For if B G be not parallel to A L, the tangent parallel to A L will meet the curve of the ellipse, or the curve of the opposite hyperbola, not in B, but in some other point, as N, as two straight lines cannot touch a conic section in the same point. Then, by the preceding Cor. if A N be drawn, it will be a diameter: which is absurd. For the

 straight

BOOK II. ſtraight line A N cannot paſs through C, the center, as C is in the diameter A B.

PROP. III.

If two parallel ſtraight lines touch an ellipſe or oppoſite hyperbolas, ſtraight lines parallel to them in the ellipſe, or in either hyperbola, will be ordinates to the diameter joining the points of contact; but, in the oppoſite hyperbolas, ſtraight lines parallel to the diameter joining the points of contact will be ordinates to the ſecond diameter parallel to the tangents; and, in either caſe, ordinates to the ſame diameter of an ellipſe, or oppoſite hyperbolas, are parallel to one another.

Fig. 37. 38. Part I. Let two parallel ſtraight lines G H, L M touch the ellipſe A D B E, or oppoſite hyperbolas F B N, I A K in the points A, B, and let F N, I K, in the ellipſe or in either hyperbola, be parallel to G H, L M; then A B will be a diameter by Cor. 1. Prop. II. and by Prop. I. it will biſect F N, I K. This part is therefore evident, by the above and the fourth Definition.

Fig. 38. Part II. The reſt remaining as above, let D E be a ſecond diameter, parallel to the tangents G H, L M, and let F I, N K, terminated by the oppoſite curves, be parallel to A B, and then by Cor. 1. Prop. II. F I, N K are biſected by D E, and are therefore ordinates to it, according to the fourth Definition.

Fig. 39. 40. Part III. Ordinates to the ſame diameter of an ellipſe, or oppoſite hyperbolas, are parallel to one another. For firſt let A B be any diameter of the ellipſe, or any tranſverſe diameter of the oppoſite hyperbolas; and C being the center, let L A, G B be the parallel tangents drawn through the vertices A, B, according to Cor. 2. Prop. II. Then, by Part I. any ſtraight line parallel

parallel to L A, or G B in the ellipſe, or in either of the oppoſite hyperbolas, will be an ordinate to A B. But, if it be poſſible, let the ſtraight line I P in the ellipſe, or in either of the oppoſite hyperbolas, be an ordinate to the diameter A B, and not be parallel to L A, G B. Draw I K parallel to L A or G B, and let it meet A B in S, and the curve in K. Draw I C, and, being produced, let it meet the curve of the ellipſe again, or the curve of the oppoſite hyperbola, in N; and draw K N. Then, by Part I. I K is biſected in S; and, as I N is biſected in C, I C : C N : : I S : S K, and therefore (2. vi.) A B, K N are parallel. Conſequently, if I P meet A B in R and K N in V, (2. vi.) I S : S K : : I R : R V, and therefore I V is biſected in R. But the ſtraight line K N in Fig. 39. is wholly within the ellipſe, and in Fig. 40. N K is without the oppoſite hyperbolas, and being produced it falls on the one ſide within one hyperbola, and on the other within the oppoſite hyperbola. The other point P therefore, in which I P meets the curve, cannot be in K N, and conſequently I P cannot be biſected in R, or be an ordinate to A B.

Fig. 40.

Laſtly, the reſt remaining as above, let D E be a ſecond diameter of the hyperbolas parallel to the tangents L A, G B. Then any ſtraight line parallel to A B, and meeting the oppoſite curves will be biſected by D E, according to Part II. But, if it be poſſible, let I X meet the oppoſite curves in I, X, the diameter D E in Y, and be an ordinate to D E, and not be parallel to A B. Let I F be parallel to A B and meet the oppoſite curves in I, F, and D E in D. Draw I C, and, being produced, let it meet the oppoſite curve in N. Draw N F, and let it meet I X in Z. Then, as I N is biſected in C, and as I F, according to Part II. is biſected in D, I C : C N : : I D : D F, and therefore (2. vi.) N F is pa-

BOOK II. rallel to D E. Confequently (2. vi.) I D : D F : : I Y : Y Z, and therefore I Y is equal to Y Z. The ftraight line I X therefore cannot be bifected in Y, or be an ordinate to D E. In every cafe therefore, ordinates to the fame diameter are parallel to one another.

Cor. 1. If a ftraight line bifect two parallel ftraight lines in an ellipfe or hyperbola, or oppofite hyperbolas, it will be a diameter: and ftraight lines drawn through its vertices, and parallel to the lines bifected, will touch the ellipfe, or oppofite hyperbolas, if in the oppofite hyperbolas it be a tranfverfe diameter. For by Cor. 2. Prop. VIII. Book I. two ftraight lines, and only two, parallel to the lines bifected, can be drawn to touch the ellipfe or oppofite hyperbolas; and by Cor. 1. Prop. II. the ftraight line joining the points of contact is a diameter, and, by this Propofition, this diameter will bifect only fuch ftraight lines in the ellipfe or oppofite hyperbolas as are parallel to the tangents. It is alfo evident from this Propofition that a ftraight line bifecting two parallel ftraight lines, terminated by the curves of oppofite hyperbolas, is a fecond diameter of the hyperbolas.

Cor. 2. If in an ellipfe, hyperbola, or oppofite hyperbolas, a diameter bifect a ftraight line not paffing through the center, it will alfo bifect any line parallel to it in the fame fection or oppofite hyperbolas. For, by the preceding Cor. a ftraight line bifecting two parallel ftraight lines in an ellipfe, hyperbola, or oppofite hyperbolas, is a diameter, and therefore paffes through the center. Confequently a diameter, or a ftraight line paffing through the center, and bifecting one of two parallel lines in the fame fection, or oppofite hyperbolas, will alfo bifect the other.

Cor. 3. A diameter of an ellipfe or hyperbola will bifect all ftraight lines in the fection parallel to a tangent paffing

passing through its vertex; and ordinates to a diameter and tangents passing through its vertices are parallel to one another. BOOK II.

PROP. IV.

Two diameters of an ellipse, or opposite hyperbolas, are conjugate diameters, if one of them be parallel to the ordinates of the other.

Let A B, D E, be two diameters of the ellipse A D B E, or of the opposite hyperbolas M A, G B F, and let the diameter D E be parallel to G F a double ordinate to A B; the diameter D E will be the conjugate diameter to A B. Fig. 41. 42.

For let C be the center, and F C being drawn, and produced, let it meet the curve of the ellipse, or the curve of the opposite hyperbola in M. Draw G M, and let it meet D E in P. Let A B meet G F in H, and then as G F is a double ordinate to the diameter A B, it is bisected in H; and as the diameter F M is bisected in C, the center, F C : C M : : F H : H G, and therefore (2. vi.) M G is parallel to A B. Again, as D E, G F are parallel, (2. vi.) M C : C F : : M P : P G, and therefore M G is bisected in P. Consequently the diameters D E, A B are conjugate to one another, according to the sixth Definition; for by Cor. 2. Prop. III. D E will bisect any straight line parallel to M G or A B, and A B will bisect any straight line parallel to G F or D E, the straight lines parallel to M G or G F being terminated by the curve of the ellipse, hyperbola, or opposite hyperbolas.

Cor. 1. From the above, Cor. 3. Prop. III. and Cor. 1. Prop. II. it is evident, that if two parallel straight lines touch an ellipse or opposite hyperbolas, and if, from any point in the curve of the ellipse, or of either hyperbola, except the points of contact, a straight

 line

line be drawn parallel to the tangents, it will be either an ordinate to the diameter joining the points of contact, or in the ellipse it will be the diameter conjugate to that joining the points of contact.

Cor. 2. Ordinates to a diameter, of an ellipse or opposite hyperbolas, tangents passing through its vertices, and its conjugate diameter are parallel to one another.

DEFINITIONS.

VIII.

Fig. 42. If C be the center of the opposite hyperbolas M A, F B, and A B a transverse diameter, to which D E is the conjugate, and H F an ordinate, and if the rectangle under A H, H B be to the square of H F as the square of C B to the square of C E or C D, the points D, E are called *the Vertices of the second Diameter* D E. In this way the magnitude of any second diameter is determined by its vertices.

Cor. If the ordinate H F be parallel to the base of the cone, in which the hyperbola was formed, the semidiameter C E or C D will be equal to the straight line drawn through C and parallel to the base of the cone, and touching either of the conical superficies. For, in this case, by the second Cor. to Prop. X. Book I. the rectangle under A H, H B is to the square of H F as the square of C B to the square of the line drawn through C and parallel to the base of the cone, and touching either of the conical superficies; and, by this Definition, the square of C B is to the square of C E or D E in the same proportion. Consequently (9. v.) the Cor. is evident, as the sides of equal squares are equal.

IX.

A straight line which is a third proportional to two conjugate diameters of an ellipse, or opposite hyperbolas,

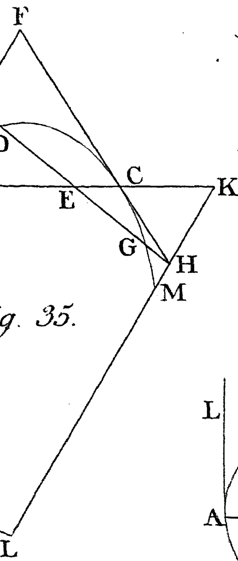


Fig. 35.

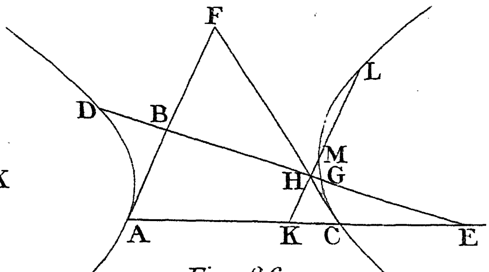


Fig. 36.

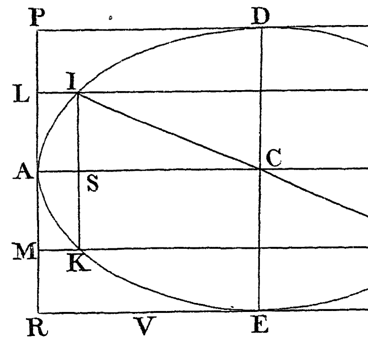


Fig. 37.

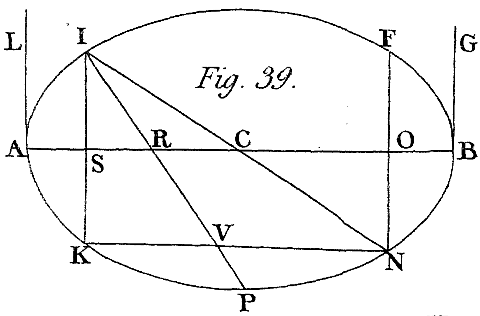


Fig. 39.

Fig. 38.

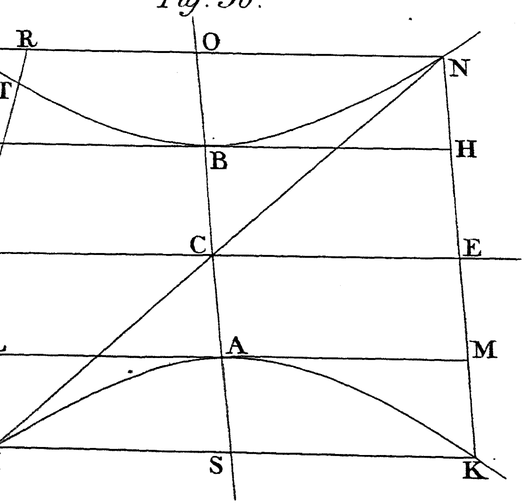
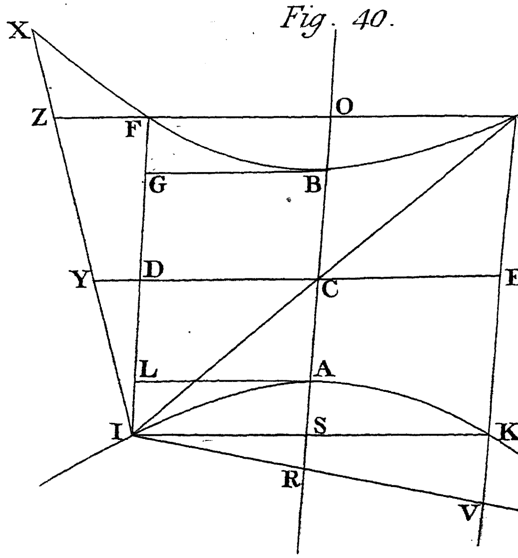


Fig. 40.



PROP. V.

If each of two straight lines, meeting one another, touch or cut, or one of them touch, and the other cut, an ellipse, hyperbola, or opposite hyperbolas; the square of the first of the two, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the second, if a tangent, or the rectangle under its segments, if a secant, as the square of the semidiameter parallel to the first to the square of the semidiameter parallel to the second.

In the ellipse, and when the two straight lines are parallel to two transverse diameters of opposite hyperbolas, the Proposition is evident from Prop. XIII. Book I. For diameters in the ellipse, and transverse diameters of opposite hyperbolas, are secants meeting one another in the center, in which they are bisected.

For other cases, first let L M, G H cutting either of the opposite hyperbolas in L, M and G, H, meet one another in K; and let C E be the semidiameter parallel to L M, and let C F be the semidiameter parallel to G H, and let C E, L M be parallel to the base of the cone in which the hyperbola was formed; and then $LK \times KM : GK \times KH :: CE^2 : CF^2$. For, bisect G H in I, and through I draw N P parallel to L M or C E, and draw the transverse diameter I B A. Then, by Cor. 2. to Prop. X. Book I. and the Cor. to Definition VIII. $NI \times IP : AI \times IB :: CE^2 : CB^2$; and by Definition VIII. $AI \times IB : IH^2 :: CB^2 : CF^2$. Consequently, Fig. 45.

$$\begin{array}{ccccc} NI \times IP & : & AI \times IB & : & IH^2 \\ CE^2 & : & CB^2 & : & CF^2, \end{array}$$

and (22. V.) $NI \times IP : IH^2 :: CE^2 : CF^2$. But, by Prop.

BOOK II. Prop. XIII. Book I. $NI \times IP : IH^2 :: LK \times KM : GK \times KH$; and therefore (11. v.) $LK \times KM : GK \times KH :: CE^2 : CF^2$.

Fig. 44. Secondly, let GH, LM cut one another in K, in the hyperbola as above, and let neither of them be parallel to the base of the cone in which the section was formed. Let GH be parallel to the semidiameter CF, and let LM be parallel to the semidiameter CE. Through K draw QR parallel to the base of the cone in which the section was formed, and let CS be the semidiameter parallel to QR, or to the base of the cone. Then, as above, we have the two following ranks of magnitudes proportionals,

$$LK \times KM : QK \times KR : GK \times KH$$
$$CE^2 \quad : \quad CS^2 \quad : \quad CF^2;$$

and therefore (22. v.) $LK \times KM : GK \times KH :: CE^2 : CF^2$.

Thirdly, the rest remaining as above, let the straight line VTK meet one of the opposite hyperbolas in V, the other in T, and the straight line QR in K. Then, by Prop. XII. Book I. and Cor. to Definition VIII. YX being the transverse diameter parallel to VT, $VK \times KT : QK \times KR :: CX^2 : CS^2$.

Lastly, every thing remaining as in the two preceding cases, by the above we have the two following ranks of magnitudes proportionals,

$$VK \times KT : QK \times KR : GK \times KH$$
$$CX^2 \quad : \quad CS^2 \quad : \quad CF^2,$$ and therefore (22. v.) $VK \times KT : GK \times KH :: CX^2 : CF^2$.

In every case therefore, by the above, and Prop. XIII. Book I. (and 11. v.) "If each of two straight lines," &c.

Cor. 1. If two straight lines be ordinates to any diameter of an ellipse, or transverse diameter of an hyperbola, the square of the first will be to the square of the second,

econd, as the rectangle under the abſciſſes correſpond-ng to the firſt to the rectangle under the abſciſſes cor-eſponding to the ſecond.

Cor. 2. From this Propoſition (and 22. vi.) it is evi-ent, that if two ſtraight lines meeting one another ouch an ellipſe, hyperbola, or oppoſite hyperbolas, hey will be to one another as the ſemidiameters to which they are parallel.

Cor. 3. From this Propoſition, and the firſt Lemma, t is evident, that if two conjugate diameters of an el-ipſe cut one another at right angles, they cannot be qual to one another; for if they were equal to one nother, the ſection would be a circle, by the firſt Lemma, as the ſquare of the ordinate would be equal o the rectangle under the correſponding abſciſſes.

PROP. VI.

f a ſtraight line be an ordinate to any diameter of an el-lipſe, or any tranſverſe diameter of an hyperbola, the rectangle under the abſciſſes of the diameter will be to the ſquare of the ordinate as the diameter to its para-meter.

Let the ſtraight line GF be an ordinate to the dia-meter AB of the ellipſe BG, or to the tranſverſe dia-meter AB of the hyperbola BG, and let BH be the pa-rameter of AB; the rectangle under the abſciſſes AF, FB is to the ſquare of FG as AB to BH. Fig. 45. 46.

Let C be the center of the ellipſe or hyperbola, and let DE be the diameter parallel to GF, and conſe-quently, by Prop. IV. the conjugate diameter to AB. Then, by the ninth Definition, $AB : DE :: DE : BH$; and therefore (Cor. 2. 20. vi.) $AB^2 : DE^2 :: AB : BH$. But (15. v.) $AB^2 : DE^2 :: CB^2 : CD^2$; and therefore (11. v.) $CB^2 : CD^2 :: AB : BH$. But, by Prop. V.

CB^2

BOOK II. $CB^2 : CD^2 :: AF \times FB : FG^2$; and confequently (11. v.) $AF \times FB : FG^2 :: AB : BH$.

Cor. 1. Let the parameter B H be at right angles to the diameter A B, and from the other vertex A, draw A H. From the point F draw F K perpendicular to A B, and let it meet A H, or A H produced, in K. Complete the rectangle K B, and it will be equal to the fquare of the ordinate F G. For, as B H, F K are at right angles to A B, they are parallel, and therefore (4. vi.) $AB : BH :: AF : FK$; and confequently, by this Propofition (and 11. v.) $AF \times FB : FG^2 :: AF : FK$. But (1. vi.) $AF : FK :: AF \times FB : FK \times FB$, and therefore $AF \times FB : FG^2 :: AF \times FB : FK \times FB$. Confequently (14. v.) $FK \times FB$ is equal to FG^2.

Cor. 2. Complete the rectangle L A B H, and let L H meet F K in M, and let K N, the fide of the rectangle K B, oppofite to B F, meet B H in N; then in the ellipfe the fquare of the ordinate F G is lefs than the rectangle under the abfcifs F B and the parameter B H, by the rectangle M N, fimilar to L B, and having one of its fides equal to B F; but in the hyperbola, the fquare of the ordinate F G, is greater than the rectangle under the abfcifs B F and the parameter B H, by the rectangle M N fimilar to L B, and having one of its fides equal to B F. This is evident from the preceding Cor.

SCHOLIUM.

On account of the deficiency of the fquare of F G from the rectangle under F B, B H in Fig. 45. Apollonius called the fection an ellipfe; and on account of the excefs of the fquare of F G above the rectangle under F B, B H in Fig. 46. he called the fection an hyperbola.

From the properties demonftrated above thefe fections

Fig. 45. and 46.) $= a$, its parameter B H $= p$, the abſciſs F B $= x$, and the ordinate F G $= y$. Then A F $=$ $a \mp x$, the negative ſign applying to the ellipſe, and the poſitive ſign to the hyperbola. And by the ſimilar triangles A B H, A F K, $a : p :: a \mp x : \frac{ap \mp px}{a} =$ FK.

Conſequently by the firſt Cor. to Prop. VI. $\frac{ap \mp px}{a} \times x$ $= px \mp \frac{px^2}{a} = px \mp \frac{p}{a}x^2 = y^2$.

PROP. VII.

If a ſtraight line, touching an ellipſe or hyperbola, meet a diameter, and from the point of contact there be drawn an ordinate to the diameter; the ſemidiameter will be a mean proportional between the ſegments of the diameter, between the center and ordinate, and between the center and tangent.

Firſt, let the ſtraight line E M, touching the ellipſe or hyperbola E I G in the point E, meet any diameter A I in the ellipſe or tranſverſe diameter of the hyperbola in M, and let E F be an ordinate to the diameter, and meet it in F, and let C be the center; the ſemidiameter C I is a mean proportional between the ſegments C F, C M. Fig. 47. 48.

For, let A B, I D be tangents paſſing through the vertices A, I, and meeting the tangent E M in B and D. Then, by Cor. to Prop. XIII. Book I. E B : E D :: A B : I D; and as, by Cor. 3. Prop. III. the ſtraight lines A B, E F, I D are parallel, it is evident (from 10. vi.) that E B : E D :: A F : I F. And as (29. i.) the triangles B A M, D I M are equiangular, A B : I D ::

A M

BOOK II. A M : I M. Hence (11. v.) A M : I M : : A F : I F, and (18. v.) A M + I M : I M : : A F + I F : I F; and by halving the antecedents it will be in the ellipse C M : I M : : C I : I F, but in the hyperbola C I : I M : : C F : I F. Consequently, by conversion, it will be in the ellipse C M : C I : : C I : C F; but in the hyperbola C I : C M : : C F : C I, and therefore C M : C I : : C I : C F.

Fig. 49. Let A I be now a second diameter of the opposite hyperbolas G K, E L and E F an ordinate to it; and let the tangent E M meet it in M, and the transverse diameter K L, parallel to E F, in N. Then, by Prop. IV. K L, A I are conjugate diameters; and therefore E P being drawn parallel to A I, and meeting K L in P, it will be an ordinate to K L. Let C be the center, and then, by the above, C P : C L : : C L : C N, and therefore (Cor. 2. 20. vi.) C P^2 : C L^2 : : C P : C N. But (34. i.) C P, E F are equal, and E P is equal to F C, and (4. vi.) E F : C N : : M F : M C, and therefore C P^2 : C L^2 : : M F : M C. Hence (17. v. and 6. ii.) K P × P L : C L^2 : : C F : M C; and (1. vi. and 11. v.) K P × P L : C L^2 : : C F^2 : C F × M C; and (16. v.) K P × P L : C F^2 or E P^2 : : C L^2 : C F × M C. But, by Def. viii. K P × P L : E P^2 : : C L^2 : C I^2, and therefore (11. and 9. v.) C F × M C is equal to C I^2, and consequently C M : C I : : C I : C F.

Fig. 47. 48. *Cor.* 1. From the above (and 17. vi.) C M × C F is equal to C I^2, and in the ellipse these equals being taken from C M^2, we have (6. ii. and 2. ii.) A M × M I equal to C M × M F. But, when A I is a transverse diameter in the hyperbola, C M^2 being taken from the equals, C M × C F, C I^2, we have (5. ii. and 3. ii.) A M × M I equal to C M × M F.

Cor. 2. As C M × C F is equal to C I^2, by taking from each C F^2 in the ellipse, we have (3. and 5. ii.)

C F

C F × F M equal to A F × F I. But A I being a tranſverſe diameter in the hyperbola, by taking the equals C M × C F, C I² from C F², we have (2. and 6. ii.) C F × F M equal to A F × F I.

Cor. 3. When A I is a diameter of the ellipſe or tranſverſe diameter of the hyperbola, by the demonſtration of the firſt part of the Propoſition, A M : I M : : A F : F I.

PROP. VIII.

If two ſtraight lines, touching an ellipſe, hyperbola, or oppoſite hyperbolas, meet one another, the diameter biſecting the line joining the points of contact will paſs through the point of concourſe.

Let the two ſtraight lines E M, G M touch the ellipſe or hyperbola E I G, or the oppoſite hyperbolas E L, G K, in the points E, G, and meet one another in M, and let the diameter C F biſect E G, the ſtraight line joining the points of contact in F; the diameter C F will paſs through M. Fig. 47. 48. 49.

For let C be the center, and let A, I be the vertices of the diameter; and then, as E G is biſected by the diameter C F, E F is an ordinate to it, and therefore, by Prop. VII. C F : C I : : C I : the ſegment of the diameter intercepted between C and the tangent E M. For the ſame reaſons C F : C I : : C I : the ſegment of the diameter intercepted between C and the tangent G M. Conſequently the ſegment of the diameter between C and the tangent E M, is equal to the ſegment of the diameter between C and the tangent G M. The diameter muſt therefore paſs through M; for if it did not, it would, upon being produced, firſt meet the one tangent, and then the other, and its ſegments between C and the tangents would be unequal.

Cor.

ing the line joining the points of contact, will be a diameter.

PROP. IX.

If two parallel straight lines touching an ellipse, or opposite hyperbolas, meet a third tangent, the rectangle under their segments, between the points of contact and the points of concourse, will be equal to the square of the semidiameter to which they are parallel; and the rectangle under the segments of the third tangent, between its point of contact and the parallel tangents, will be equal to the square of the semidiameter to which it is parallel.

Fig. 50. 51. 52. Let the parallel straight lines A B, I D touch an ellipse E I, or opposite hyperbolas A, I, in the points A, I, and let them meet in B, D a straight line B D, which touches the ellipse, or one of the opposite hyperbolas in E; then the rectangle under the segments A B, I D is equal to the square of the semidiameter parallel to A B or I D, and the rectangle under the segments B E, E D is equal to the square of the semidiameter parallel to B D.

For let C be the center, and draw E G parallel to A B, I D; and draw also A I. Then by Cor. 1. to Prop. II. A I is a diameter, and, by Cor. 1. to Prop. IV. E G is either an ordinate to A I, or in the ellipse the conjugate diameter to it.

Fig. 50. First, let E G be the conjugate diameter to A I, and then by Prop. IV. and Cor. 3. Prop. III. B D, A I are parallel to one another, and also A B, C E, I D to one another. Consequently (34. i.) A B, C E, I D are equal to one another, as are also A C, C I, B E, E D to one another;

Fig. 41.

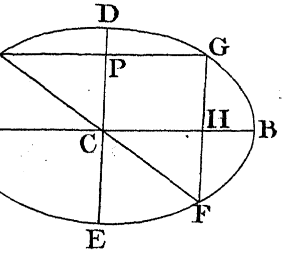


Fig. 42.

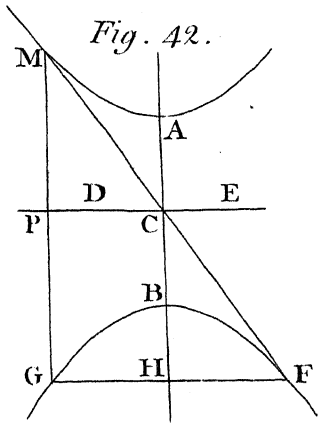


Fig. 43.

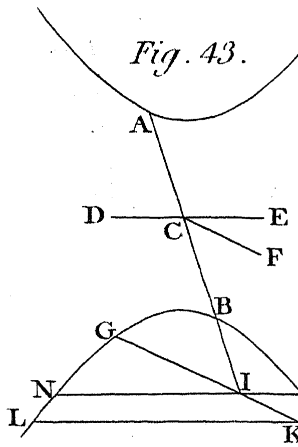


Fig. 44.

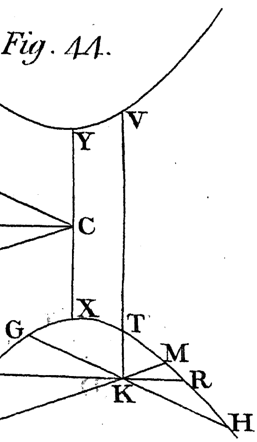


Fig. 45.

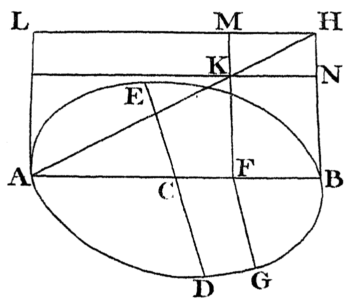


Fig. 46.

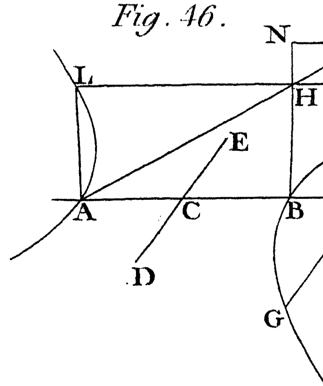


Fig. 47.

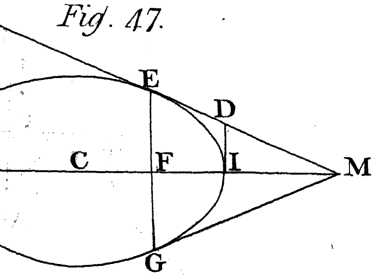


Fig. 48.

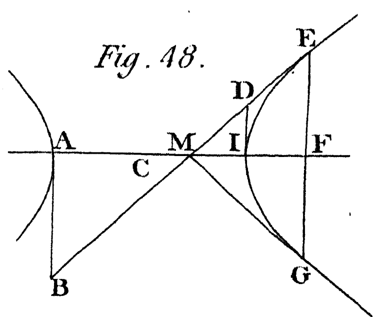
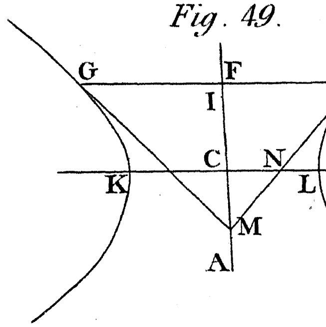


Fig. 49.



another; and therefore A B × I D is equal to C E^2, and B E × E D is equal to A C^2.

Fig. 51. 52.

Next, let E G be an ordinate to the diameter A I, and let it meet it in F. Let C K be the ſemidiameter parallel to the tangents A B, I D, and C P the ſemidiameter parallel to the tangent E B; and let E B meet the diameter A I in M, and the diameter K C L in H. Let E O be an ordinate to K C L, and let it meet it in O. Then, by Cor. 3. Prop. III. and Prop. IV. A B, L H, G E, I D are parallel, and E O is parallel to A I. By Cor. 1. Prop. VII. A M × M I is equal to C M × M F, and therefore (16. vi.) A M : C M : : M F : M I; and (4. vi.) A M : C M : : A B : C H, and M F : M I : : F E or C O : I D. Conſequently (11. v.) A B : C H : : C O : I D, and (16. vi.) A B × I D is equal to C H × C O, which is equal to C K^2, by Prop. VII. (and 17. vi.) and therefore A B × I D is equal to C K^2.

Again, by Cor. 2. Prop. V. A B : B E : : C K : C P, and by Cor. to Prop. XIII. Book I. A B : I D : : B E : E D. Hence (22. vi.) A B × I D : B E × E D : : C K^2 : C P^2, and as by the above A B × I D is equal to C K^2, it follows (14. v.) that B E × E D is equal to C P^2.

Cor. 1. Every thing remaining as above, as, by Cor. 2. Prop. VII. A F × F I is equal to C F × F M, (16. vi.) A F : F M : : C F : F I. But on account of the parallels (10. vi.) A F : F M : : B E : E M; and C F : F I : : H E : E D. Conſequently (11. v.) B E : E M : : H E : E D, and therefore (16. vi.) M E × E H is equal to B E × E D or to C P^2.

Cor. 2. If a ſtraight line as B E, touching an ellipſe or hyperbola in the point E, meet the diameter A I in M, and the diameter K L in H, and if the rectangle under M E, E H be equal to the ſquare of the ſemidiameter parallel to B E, the diameters A I, K L will be conjugate to one another.

BOOK II.

PROP. X.

To find the axes of a given ellipse or hyperbola, the center being also given; and to demonstrate that the same section can have only two axes.

Fig. 53. 54. Part I. Let D B E be a given ellipse or hyperbola, of which C the center is also given; it is required to find the axes of the section.

Fig. 53. In the ellipse draw C G, C F two semidiameters, and, if they be unequal, let C G be greater than C F. With C as a center, and a distance less than C G but greater than C F, describe the circle H D E. Then from this construction and the nature of the two curves it is evident, that the circumference of the circle will cut the curve of the ellipse in four points, two of them being towards the left of the center, as the figure is viewed, and two of them towards the right. Let the circumference of the circle cut the curve of the ellipse in the points D, E. Draw the straight line D E; and through C draw A B bisecting D E in I, and (3. iii.) A B will be at right angles to D E. Through C draw L M parallel to D E, and A B, L M will be the axes of the ellipse.

For, by construction, D E is an ordinate to the diameter A B, and at right angles to it. Again, as L M is parallel to D E, and as A I D is a right angle, the angle A C L (29. i.) is also a right one. By Prop. IV. the diameter L M will also bisect all straight lines in the ellipse parallel to A B, and it will therefore cut its ordinates at right angles. Consequently A B, L M are axes of the ellipse, according to the seventh Definition.

If the semidiameters C G, C F be equal, then a diameter bisecting the angle G C F will be one of the axes, and a diameter at right angles to it will be the other. For, in this case, if G F be drawn, it will be bisected at right

right angles (4. i.) by the diameter bisecting the angle G C F; and the rest will be as above.

Next, let the section D B E be an hyperbola, of which C is the center, and let K be any point within the hyperbola. With C as a center, and C K as a distance, describe the circle E K D, and let its circumference cut the curve of the hyperbola in the points E, D. Draw D E and bisect it in I, and through I draw the diameter A B; and parallel to D E draw the diameter L M. The diameters A B, L M are the axes of the hyperbola. For D E is a double ordinate to the diameter A B, and (3. iii.) A B cuts it at right angles, and L M is parallel to the ordinate D E. Fig. 54.

Part II. To demonstrate that an ellipse or hyperbola can have only two axes. First, let the section B D A be an ellipse, and C being the center, let R L, F G be the axes, found as above; and, if it be possible, let the diameter A B be also an axis. Let L D be a double ordinate to A B, meeting it in E, and the curve again in D; and the diameter D K being drawn, let D H be an ordinate to R L*. Then, as by hypothesis A B is an axis, C E L, C E D are right angles, and as L D is bisected in E, (4. i.) C L is equal to C D. Again, as by the above R L, F G are conjugate, D H is parallel to F G, by Prop. IV. and by Prop. V. $CL^2 : CF^2 :: RH \times HL : DH^2$. But, by Cor. 3. Prop. V. C L, C F must be unequal, and therefore, supposing C L to be the greatest, CL^2 is greater than CF^2, and $RH \times HL$ greater than DH^2. To these unequals add CH^2, and then (5. ii. and 47. i.) CL^2 is greater than CD^2; and consequently C L is greater than C D. But C D is also equal to C L: which Fig. 55.

* A method of drawing an ordinate to a given diameter will be inserted hereafter. The insertion of it previous to the above, or at this place, would have caused a needless repetition.

BOOK II. is abſurd. The diameter A B therefore cannot be an axis.

Fig. 56. Next, let the ſection E B D be an hyperbola, of which C is the center, F A the oppoſite hyperbola, and A B, L M the axes found as above; and, if it be poſſible, let the tranſverſe diameter D F be an axis. Let D T touch the hyperbola in D, and meet the axis A B in T; and let D I E be an ordinate to A B, and let it meet it in I. Then in the triangle C I D, C I D is a right angle, by Def. VII. and therefore C D I is leſs than a right angle, and conſequently C D T is much leſs than a right angle. But, by Cor. 2. Prop. IV. the tangent T D is parallel to the ordinates of the axis F D, and therefore, by Def. VII. (and 29. i.) the angle T D C is a right one. And, by the above, it is alſo leſs than a right one: which is abſurd. Conſequently D F is not an axis. Nor can a ſecond diameter as G H, beſides L M, be an axis. For F D conjugate to G H being drawn, and D T a tangent, the demonſtration would end in the ſame abſurdity.

Cor. From the above, and Prop. IV. it is evident, that the axes of an ellipſe, or hyperbola, are conjugate diameters.

PROP. XI.

Of all the diameters of an ellipſe the greater axis is the greateſt and the leſſer axis is the leaſt; and of oppoſite hyperbolas the axes are the leaſt diameters.

Fig. 55. Part I. Let A B D be an ellipſe, of which C is the center, R L the greater axis, and F G the leſſer axis; of all the diameters R L is the greateſt and F G the leaſt.

For let K D be any other diameter, and let D H be an ordinate to R L, and D M an ordinate to F G. Then, by Cor. 2. Prop. IV. D H is parallel to F G, and D M to

R L;

R L; and therefore, by Prop. V. $CL^2 : CF^2 :: RH \times HL : DH^2$, and as C L is greater than C F, CL^2 is greater than CF^2, and $RH \times HL$ is greater than DH^2. To these add CH^2, and (5. ii. and 47. i.) then CL^2 is greater than CD^2. Consequently C L is greater than C D, and therefore R L is greater than K D. Again, by Prop. V. $CL^2 : CF^2 :: DM^2 : GM \times MF$, and therefore, as above, DM^2 is greater than $GM \times MF$. To these add the square of C M, and then (5. ii. and 47. i.) CD^2 is greater than CF^2. Consequently K D is greater than F G.

Part II. Let A B, L M be the axes of the opposite hyperbolas E B D, A F, and F D, G H any other conjugate diameters; then A B is less than the transverse diameter F D, and L M is less than G H. Fig. 56.

For let D E be an ordinate to the axis A B, and let it meet it in I, and let D T touch the hyperbola in D, and let it meet A B in T. Let B P touch the hyperbola in the vertex B, and let it meet the tangent D T in P, and let C be the center. Then, by Def. VII. C I D is a right angle, and therefore (19. i.) C D is greater than C I, and consequently much greater than C B. Hence the transverse axis is less than the transverse diameter F D. Again, by Cor. 2. Prop. IV. G H is parallel to T D, and D I to P B; and, by Prop. VII. C I : C B :: C B : C T, and therefore by conversion, C I : B I :: C B : B T. Hence, as C I is greater than C B, (14. v.) B I is greater than B T. But (2. vi.) B I : B T :: D P : P T, and therefore D P is greater than P T; and, as (29. i.) P B T is a right angle, P T (19. i.) is greater than B P. Hence D P is greater than B P; and as, by Cor. 2. Prop. V. D P : B P :: C H : C L, C H is greater (14. v.) than C L. Consequently the axis L M is less than the second diameter G H.

BOOK II.

DEFINITIONS.

X.

Fig. 60. 61. In the ellipſe the greater axis is called the *Tranſverſe Axis*, and the other axis is called the *Conjugate Axis*; and in the hyperbola the axis which is a tranſverſe diameter is called the *Tranſverſe Axis*, and the other axis is called the *Conjugate Axis*.

XI.

If C be the center, A B the tranſverſe axis, and D E the conjugate axis of the ellipſe A D B, or of the oppoſite hyperbolas A I, B P, then if in A B two points F, O be ſo taken that the rectangle under A F, F B, and alſo the rectangle under A O, O B, be equal to the ſquare of C D or C E, the ſemiconjugate axis; the points F, O are called the *Foci*, or *Umbilici*, of the ellipſe, hyperbola, or oppoſite hyperbolas.

Cor. 1. As (axiom i. 1.) A F × F B is equal to A O × O B, the foci F, O are equally diſtant from the vertices A, B, by the ſixth Lemma. It is alſo evident, that the foci are equally diſtant from the center.

Fig. 60. *Cor.* 2. In the ellipſe the diſtance of each of the foci from either extremity of the conjugate axis is equal to the ſemitranſverſe axis. For, ſuppoſing a ſtraight line to be drawn from D to O, the ſquare of D O (47. i.) will be equal to the ſquares of C O, C D together; and therefore, by this Definition, (and 5. ii.) the ſquare of D O is equal to the ſquare of A C. Conſequently D O is equal to A C; and therefore if with D or E as a center, and A C or C B as a diſtance, a circle be deſcribed, the circumference will cut A B in O, and F the foci.

Fig. 61. *Cor.* 3. In the hyperbola the diſtance of each of the foci from the center is equal to the diſtance between the vertices of the tranſverſe and conjugate axes. For, ſuppoſing a ſtraight line to be drawn from D to A, the ſquare of D A (47. i.) will be equal to the ſquares of C A,

C A, C D together; and therefore, by this Definition, (and 6. ii.) the ſquare of D A is equal to the ſquare of C O or C F. Conſequently C O or C F is equal to D A; and therefore the foci F, O may be eaſily found from the axes.

Cor. 4. The double ordinate T S to the axis A B, drawn through either focus, ſuppoſe F, is equal to the parameter of the axis A B. For, by Prop. V. $CB^2 : CD^2 :: AF \times FB$, or by this Def. as $CD^2 : TF^2$; and therefore (22. vi.) C B : C D :: C D : T F. Conſequently (15. 5.) A B : D E :: D E : T S, and the Cor. is evident from Def. IX. Fig. 60. 61.

XII.

If through the foci F, O of an ellipſe A D B, or of the oppoſite hyperbolas A I, B P, ordinates F T, O I to the axis A B be drawn, and if through the points T, I in which they meet the curve ſtraight lines H T, I L be drawn to touch the ſection, or oppoſite hyperbolas, the tangents H T, I L are called *Focal Tangents*.

PROP. XII.

If a tangent, paſſing through a vertex of the tranſverſe axis of an ellipſe or oppoſite hyperbolas, meet a focal tangent, its ſegment between the point of contact and point of concourſe will be equal to the ſegment of the axis between the point of contact and the focus, to which the focal tangent belongs.

Let T B be an ellipſe or hyperbola, of which A B is the tranſverſe axis, and F, O the foci, and let A H touch the ellipſe, or either of the oppoſite hyperbolas, in the vertex A, and meet in the point H the focal tangent H G, belonging to the focus F; the ſegment A H is equal to the ſegment A F. Fig. 60. 61.

For let H G touch the ſection in T, and T F, being

 drawn,

BOOK II. drawn, will be an ordinate to A B, by the twelfth Definition. Let B G touch the section in the vertex B, and meet H G in G; and let C be the center, and D E the conjugate axis. Then, by Cor. 2. Prop. IV. A H, D E, T F, B G are parallel; and therefore, by Cor. to Prop. XIII. Book I. A H : B G : : H T : T G. But it is evident, (from 10. vi.) that H T : T G : : A F : F B; and therefore (11. v.) A H : B G : : A F : F B. Consequently A H × B G is similar to A F × F B, and, by Prop. IX. and Def. XI. each of these rectangles is equal to the square of C D. They are therefore equal to one another; and, as they are also similar, A H is equal to A F, and B G is equal to B F.

Cor. If O I be drawn an ordinate to A B, and on the side of A B opposite to that on which F T is, and if through I, the point in which it meets the curve, there be drawn the focal tangent K L, meeting the tangent H A in K, and the tangent G B in L; then H L will be a parallelogram, and each of the opposite sides H K, G L will be equal to the transverse axis A B. For, by the above, and Cor. 1. to Def. XI. A K, A O, B F, B G are equal to one another, and also A H, A F, B O, B L to one another. Consequently H K, G L are equal and parallel, and therefore (33. i.) H G, K L are equal and parallel. Hence H L is a parallelogram; and as A H is equal to A F, and A K to B F, H K or G L is equal to A B.

PROP. XIII.

If from any point in the curve of an ellipse or hyperbola two straight lines be drawn to the foci, their sum in the ellipse, but their difference in the hyperbola, will be equal to the transverse axis.

Fig. 60. 61. Let P be any point in the curve of the ellipse, or hyperbola P T B, of which A B is the transverse axis, and

and the points F, O the foci, and let P O, P F, be ſtraight lines drawn to the foci; the ſum of P O, P F in the ellipſe, but their difference in the hyperbola, is equal to A B.

For, the reſt remaining as in the preceding Propoſition and its Corollary, let P R be drawn an ordinate to A B, and let it meet the curve again in M, the focal tangent G H in N, and the focal tangent K L in Q. Then, by Cor. 2. Prop. IV. H K, O I, D E, N Q, T F, G L are parallel, and therefore, by Prop. XIII. Book I. $TH^2 : TN^2 :: AH^2 : MN \times NP$. But, on account of the parallels (10. vi.) $TH^2 : TN^2 :: FA^2 : FR^2$; and therefore (11. v.) $FA^2 : FR^2 :: AH^2 : MN \times NP$, and as, by Prop. XII. F A is equal to A H, FA^2 is equal to AH^2, and therefore FR^2 is equal to $MN \times NP$. To theſe equals add the ſquare of P R, and then (6. ii. and 47. i.) the ſquare of N R is equal to the ſquare of P F, and conſequently N R is equal to P F. Again, by Prop. XIII. Book I. $IL^2 : IQ^2 :: BL^2 : PQ \times QM$; and on account of the parallels (10. vi.) $IL^2 : IQ^2 :: OB^2 : OR^2$, and therefore (11. v.) $OB^2 : OR^2 :: BL^2 : PQ \times QM$. But, by Prop. XII. O B is equal to B L, and therefore OB^2 is equal to BL^2, and (14. v.) OR^2 is equal to $PQ \times QM$. To theſe equals add the ſquare of R M, or its equal the ſquare of R P, and then (6. ii. and 47. i.) the ſquare of O M in the ellipſe, and the ſquare of O P in the hyperbola, is equal to the ſquare of R Q. Conſequently in the hyperbola O P is equal to R Q, and in the ellipſe O M is equal to R Q. But in the ellipſe P R, R M are equal, and the angles O R M, O R P are equal, being right angles, and O R is common to the two triangles O R M, O R P, and therefore (4. i.) O M is equal to O P. In each ſection therefore O P is equal to R Q, and P F to N R. Conſequently in the ellipſe the ſum of P O, P F, but in the hyperbola their dif-

BOOK II. difference is equal to N Q. But Q H is a parallelogram, and therefore (34. i.) N Q, H K are equal; and as, by the Cor. to Prop. XII. H K is equal to A B, the sum of P O, P F in the ellipse, but their difference in the hyperbola is equal to A B, the transverse axis.

Cor. 1. If from any point in the curve of an ellipse, or hyperbola, two straight lines be drawn to the foci, in the ellipse the difference between the transverse axis and either of the two will be equal to the other; but in the hyperbola the sum of the transverse axis and the least of the two will be equal to the other.

Cor. 2. If the conjugate axis D E be produced till it meet the opposite focal tangents in V and W, each of the segments C V, C W between C the center and a focal tangent will be equal to the semitransverse axis. For, let the opposite focal tangents meet the transverse axis A B in X and Y. Then, as, by Cor. 1. Def. XI. C F, C O are equal, it is evident, from Prop. VII. that C Y, C X are equal; and as X W, V Y are parallel, the angles (29. i.) C X W, C Y V are equal. Consequently, as the angles at C are right angles, (26. i.) C V is equal to C W. In each section therefore the Cor. is evident.

PROP. XIV.

If from a point without an ellipse or opposite hyperbolas two straight lines be drawn to the foci, their sum in the ellipse will be greater, but their difference in the hyperbola will be less, than the transverse axis. But if from a point within an ellipse or hyperbola two straight lines be drawn to the foci, their sum in the ellipse will be less, but their difference in the hyperbola greater, than the transverse axis.

Fig. 57. 58. Part I. Let E be a point without the ellipse A D B, or opposite hyperbolas A, B, and let E F, E O be straight lines

lines drawn to the foci F, O; the ſum of E F, E O in the ellipſe is greater, but their difference in the hyperbolas leſs, than A B the tranſverſe axis.

In the ellipſe let E F cut the curve in D, and draw O D; and then (20. i.) O E, E D together being greater than O D, the three O E, E D, D F together are greater than O D, D F together. Conſequently, by Prop. XIII. the ſum of E F, E O is greater than A B. In the hyperbolas let E F be greater than E O, and let E O cut the curve of the hyperbola A in D, and draw F D. Then D F, D E together are greater than E F. But, by the Cor. to Prop. XIII. D F is equal to A B, O D together, and therefore A B, O D, D E together, or A B and O E together, are greater than E F. Conſequently the difference between E F, E O is leſs than A B. Fig. 57. Fig. 58.

Part II. Let G be a point within the ellipſe or hyperbola A D, and let the ſtraight lines G F, G O be drawn to the foci F, O; the ſum of G F, G O in the ellipſe is leſs, but in the hyperbola their difference is greater, than A B the tranſverſe axis.

In either ſection let G F meet the curve in D, and draw D O. Then, in the ellipſe, O D, D G together (20. i.) are greater than O G; and therefore O D, D G, G F together, or O D, D F together, are greater than O G, G F together. Conſequently the ſum of G F, G O is leſs than A B, by Prop. XIII. In the hyperbola O D, D G together (20. i.) are greater than G O, and therefore O D, D G, A B together are greater than G O, A B together. But, by Cor. 1. to Prop. XIII. O D, A B together are equal to D F, and therefore D F, D G together, or F G, are greater than G O, A B together. Conſequently the difference between F G, G O is greater than A B.

Cor. 1. From this and Prop. XIII. it is evident, that two ſtraight lines being drawn from a point to the foci of

BOOK II. of an ellipfe or hyperbola, if in the ellipfe their fum be greater, or in the hyperbola their difference be lefs, than the tranfverfe axis, the point will be without the fection. If in the ellipfe the fum of the two lines, or in the hyperbola their difference, be equal to the tranfverfe axis, the point will be in the curve of the fection. Laftly, if in the ellipfe the fum of the two lines be lefs, or in the hyperbola their difference be greater, than the tranfverfe axis, the point will be within the fection.

Fig. 59. *Cor.* 2. If O, F be the foci of the hyperbola B P, and if the fide O D, of the triangle O D F, be equal to A B, the tranfverfe axis, and O D F be an obtufe angle, then the ftraight line O D produced will meet the curve of the hyperbola B P, in which the focus F is fituated. For let O D be produced to K, and make the angle D F L equal to the angle F D K. Then, as by hypothefis O D F is an obtufe angle, K D F is an acute angle, and therefore, as the angle D F L is equal to it, the ftraight lines D K, F L, being produced, will meet. Let them meet in P, and (6. i.) P F will be equal to P D. Confequently, as the difference of P O, P F is equal to O D, or A B, the point P is in the curve of the hyperbola, by the preceding Corollary.

PROP. XV.

If from any point in the curve of an ellipfe, or hyperbola, two ftraight lines be drawn to the foci, the ftraight line bifecting the angle adjacent to that contained by them will touch the ellipfe; but the ftraight line bifecting the angle contained by them will touch the hyperbola.

Fig. 62. 63. From the point P, in the curve of the ellipfe or hyperbola B P, let two ftraight lines P F, P O, be drawn to the foci F, O, and in the ellipfe let O P be produced to D; the ftraight line P E bifecting the angle F P D, ad-

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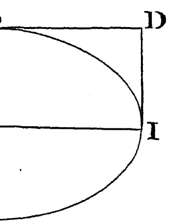


Fig. 51.

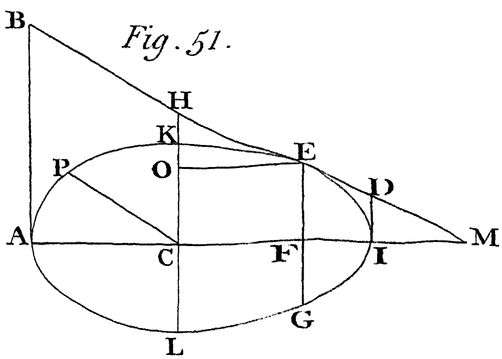
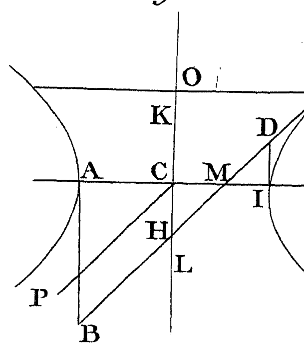


Fig. 52.



g. 53.

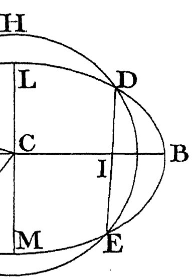


Fig. 58.

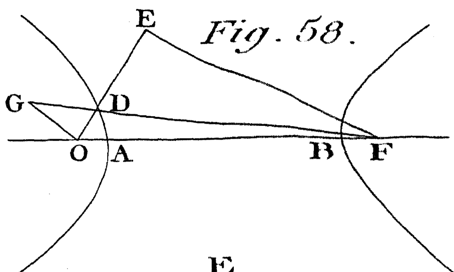


Fig. 56.

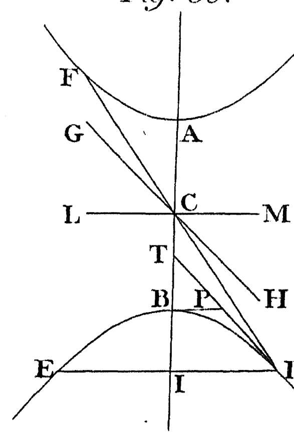


Fig. 54.

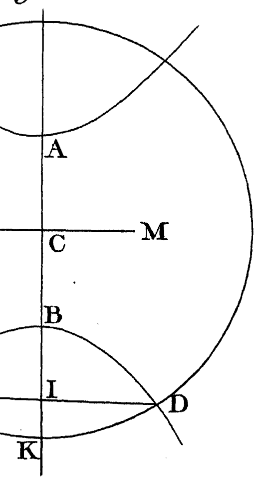


Fig. 57.

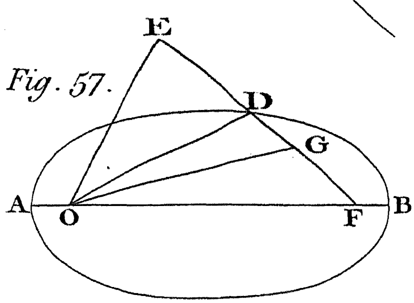


Fig. 59.

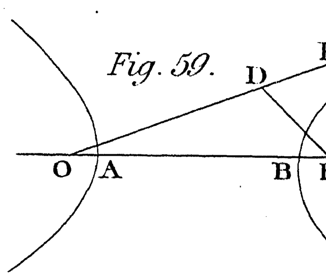
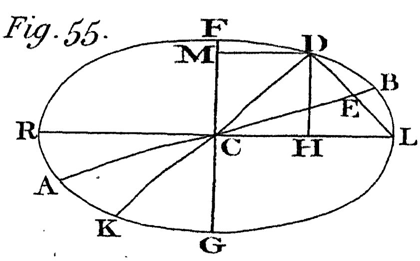


Fig. 55.



adjacent to F P O in the ellipſe, touches the ellipſe; but the ſtraight line P E, biſecting the angle F P O in the hyperbola, will touch the hyperbola.

In the ellipſe let P D, the part of O P produced, be equal to P F. Draw F D, and let it meet P E in E. Then, as F P, P D are equal, and P E common to the two triangles P E F, P E D, and the angle F P E equal to the angle D P E, the ſide F E (4. i.) is equal to D E, and the angles F E P, D E P are equal. In E P take any point G, and draw G O, G F, G D. Then, as F E is equal to E D, and as the angles F E G, D E G are equal, we have F G (4. i.) equal to D G. But (20. i.) D G, G O together are greater than D O, or O P, P F together; and therefore, by Prop. XIII. D G, G O together are greater than A B the tranſverſe axis. Conſequently G O, G F together are greater than A B, and therefore, by Cor. 1. to Prop. XIV. the point G is without the ellipſe B P, and conſequently P E touches it in P. Fig. 62.

In the hyperbola take P D in P O equal to P F. Draw F D, and let it meet P E in E. Then, as P F, P D are equal, and as P E is common to the two triangles F P E, D P E, and as the angles F P E, D P E are equal, the ſide F E (4. i.) is equal to the ſide E D, and the angles F E P, D E P are alſo equal. In E P take any point G, and draw G O, G D, G F. Then, as F E, E D are equal, and as the angles F E G, D E G are alſo equal, and E G common to the two triangles F E G, D E G, the ſide F G (4. i.) is equal to the ſide D G. Alſo, as P D is equal to P F, by Prop. XIII. D O is equal to A B, the tranſverſe axis. But G D, D O together (20. i.) are greater than G O, and therefore G F and A B together are greater than G O. Conſequently the difference between G O and O F is leſs than A B, and therefore, by Cor. 1. to Prop. XIV. the point G is without the hyperbola, and P E touches the hyperbola. Fig. 63.

Cor.

BOOK II.

Cor. 1. From this Prop. and Prop. VI. Book I. it is evident, that if a ſtraight line touch an ellipſe or hyperbola, and ſtraight lines be drawn from the point of contact to the foci, in the ellipſe the tangent will biſect the angle adjacent to that contained by theſe two ſtraight lines drawn to the foci; but in the hyperbola the tangent will biſect the angle contained by theſe two ſtraight lines drawn to the foci. In the ellipſe the angle O P G (15. i.) is equal to the angle F P E.

Fig. 62.

Cor. 2. If from the foci O, F of an ellipſe, or hyperbola, two ſtraight lines, O D, F D be drawn to a third point D, of which O D, one of them, is equal to the tranſverſe axis A B, and if the other F D be biſected in E, by a ſtraight line P E at right angles; the perpendicular P E will ſomewhere touch the ſection, provided, in the hyperbola, O D F be an obtuſe angle. And, on the contrary, if P E touch the ſection and biſect F D in E at right angles, then O D will be equal to the tranſverſe axis. This is evident from Cor. 2. Prop. XIV. Prop. VI. Book I. and the above demonſtration.

Fig. 62. 63.

Cor. 3. The reſt remaining as above, let F G be at right angles to the ſtraight line L G, touching the ellipſe or hyperbola in L, and let F G be produced to H, ſo that G H may be equal to F G; then a ſtraight line, biſecting D H at right angles, will paſs through the focus O. For, by the preceding Cor. O H is equal to the tranſverſe axis, and conſequently equal to O D. If therefore O K be drawn, biſecting D H in K, the angles (8. i.) O K D, O K H will be equal. Hence the Cor. is evident.

Fig. 64. 65.

Cor. 4. The reſt remaining as in the demonſtration of the Propoſition, let D F be ſo divided in L, that D L may be to L F as A B to O F, or, which is the ſame thing, as D O to O F, and in D F, produced as in the figures, let D N be to N F as A B to O F, and then a circle deſcribed upon

Fig. 66. 67.

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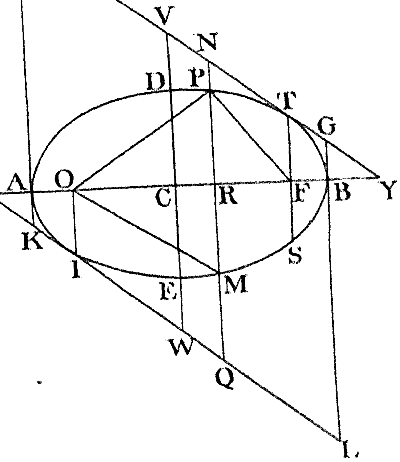
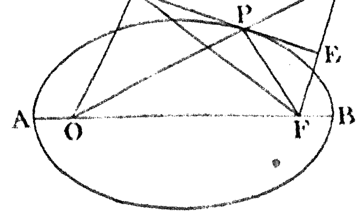




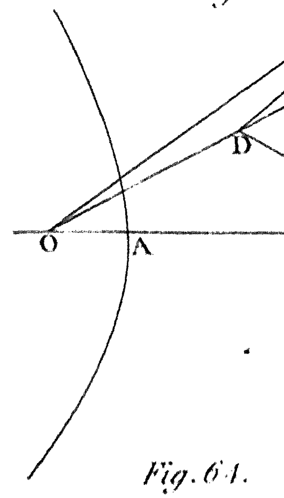


Fig. 61.

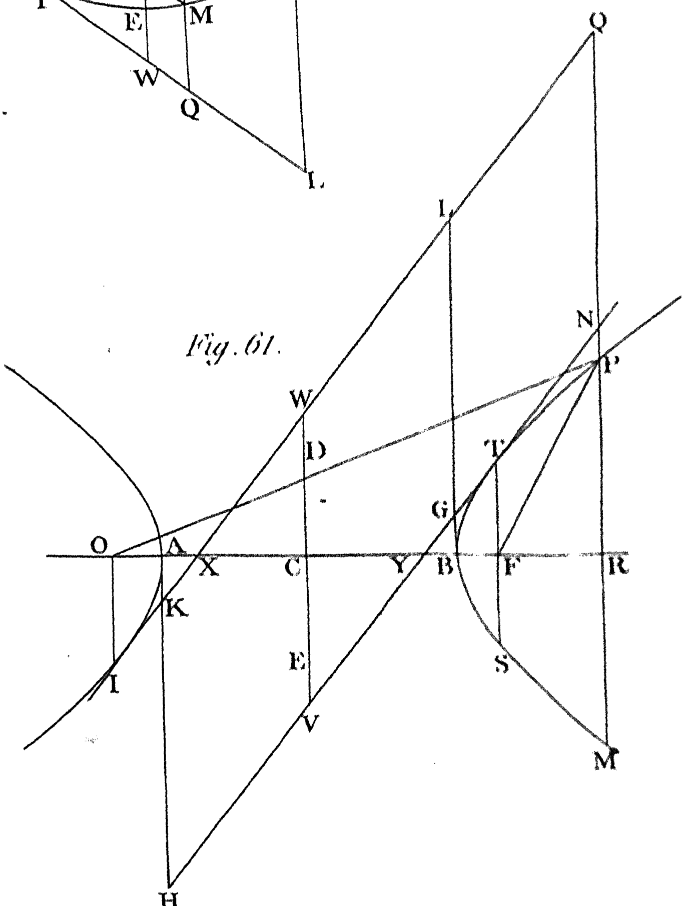


Fig. 64.

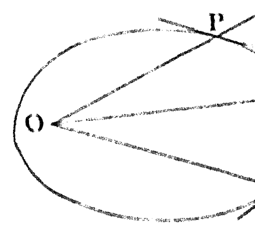
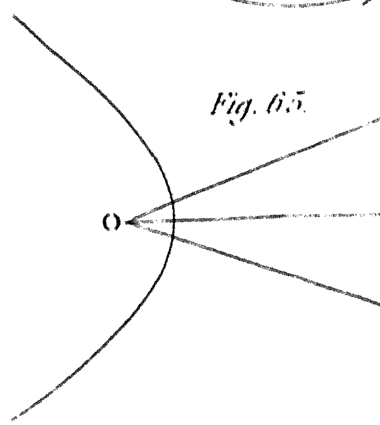


Fig. 65.



L N as a diameter will paſs through the focus O. For in the ellipſe produce D O to M, but in the hyperbola produce F O to M, and draw O L, O N. Then the angles D O F, F O M* (3. vi.) in the ellipſe are biſected by the ſtraight lines O L, O N; and as the angles D O F, F O M together (13. i.) are equal to two right angles, the angles L O F, F O N together are equal to a right angle. But in the hyperbola the angles D O F, D O M* are biſected (3. vi.) by the ſtraight lines O L, O N; and as the angles D O F, D O M together are equal to two right angles, the angles L O D, D O N together are equal to a right angle. In either caſe therefore the angle L O N is a right angle, and conſequently (31. iii. and 21. i.) a circle deſcribed about L N as a diameter will paſs through O.

Sir Iſaac Newton makes much uſe of the properties expreſſed in the three laſt Corollaries. See the Principia, Sect. IV. Book I.

PROP. XVI.

If a ſtraight line touching an ellipſe or hyperbola meet a ſtraight line drawn from either of the foci, and be at right angles to it, the ſtraight line joining the center and the point of concourſe will be equal to the ſemitranſverſe axis: or, if a ſtraight line touch an ellipſe or hyperbola, and ſtraight lines be drawn from the point of contact to the foci, a ſtraight line drawn from the center to the tangent, and parallel to either of the two drawn to the foci, will be equal to the ſemitranſverſe axis.

Part I. Let the ſtraight line G P, touching the ellipſe or hyperbola P B in the point P, meet in the points G, Fig. 68. 69.

* That the angle F O M in the ellipſe, and the angle D O M in the hyperbola, is biſected by O N, is proved by Simſon, in his Prop. A. in the ſixth Book of his edition of Euclid.

E, the

draw C G, C E to the points of concourse; each of the straight lines C G, C E is equal to A C or C B, the semi-transverse axis.

For, draw F P, O P, and let F P, or F P produced, meet O E produced in D. Then, as P E O is a right angle, P E D is a right angle; and as P E is common to the two triangles P E O, P E D, and as, by Cor. 1. to Prop. XV. the angles O P E, D P E are equal, the side P D (26. i.) is equal to the side P O, and D E is equal to E O. Consequently, by Prop. XIII. F D is equal to the transverse axis A B; and as F C, C O are equal, and D E equal to E O, F O : C O : : D O : E O. The straight lines F D, (2. vi.) C E are therefore parallel, and O F : F D : : O C : C E. But O C is the half of O F, and therefore C E is equal to the half of F D. Consequently C E is equal to A C or C B, the semitransverse axis. If F G, P O be produced till they meet in H, it may be proved, in the same manner, that F G is equal to G H, F P to P H, H O to A B, and C G to A C or C B.

Fig. 68. 69. Part II. Let the straight line G P touch the ellipse or hyperbola P B in the point P, and let P F, P O be straight lines drawn from the point of contact to the foci F, O. Let C be the center, and draw C E parallel to F P, and C G parallel to O P, and let C E meet the tangent in E, and C G meet it in G; each of the straight lines C E, C G is equal to A C or C B, the semi-transverse axis.

For draw O E, and, being produced, let it meet F P, or F P produced, in D. Then, as F D, C E are parallel, (2. vi.) F C : C O : : D E : E O, and as, by Cor. to Def. XI. F C, C O are equal, D E is equal to E O. By Cor. 1. to Prop. XV. the angles O P E, D P E are equal, and therefore (3. vi.) O E : E D : : O P : P D, and as

O E,

O E, E D are equal, O P is equal to P D. Confequently, by Prop. XIII. F D is equal to A B, the tranfverfe axis; and as F D, C E are parallel, F O : F D :: C O : C E. But C O is the half of F O, and therefore C E is the half of F D. Confequently C E is equal to A C or C B, the femitranfverfe axis. If F G be drawn, and, being produced, meet O P produced in H, it may be proved, in the fame manner, that F G is equal to G H, F P to P H, and C G to A C or C B.

Cor. The reft remaining as above, if the ftraight line I K, drawn through C the center, and parallel to the tangent G P, meet P F in I and P O in K, the fegments P I, P K are equal; and each of them is equal to A C or C B. For (34. i.) P K is equal to C G and P I is equal to C E.

The demonftrations of the 11th and 12th Propofitions of the firft Book of the Principia depend, in a very confiderable degree, on this property.

PROP. XVII.

The rectangle contained under two ftraight lines, drawn from the foci of an ellipfe or hyperbola to a tangent, and at right angles to it, is equal to the fquare of the femiconjugate axis. And the rectangle contained under two ftraight lines, drawn from the tranfverfe axis of an ellipfe or hyperbola to a tangent, and at right angles to it, is equal to the fquare of the femiconjugate axis, if one of them be drawn from the center, and the other meet the tangent in the point of contact.

Part I. Let the ftraight lines F G, O E, drawn from the foci F, O of the ellipfe or hyperbola P B, meet in the points G, E, the ftraight line G E which touches the fection in P, and let them be at right angles to the tangent G E, and C being the center, let C D be the femiconjugate Fig. 70. 71.

miconjugate

BOOK II.

miconjugate axis; the rectangle under F G, O E is equal to the square of C D.

Draw E C, and, being produced, let it meet F G, or F G produced, in H. Then as F G, O E are at right angles to G P E, they are parallel to one another, (28. i.) and (29. i.) the angles C F H, C O E are equal. The triangles (15. and 32. i.) C F H, C O E are therefore equiangular, and, Cor. 1. to Def. XI. C F is equal to C O. Consequently (26. i.) C H is equal to C E, and F H is equal to O E. If therefore, with C as a center, and C A or C B as a distance, a circle be described, it will pass through the points E, G, by Prop. XVI. and consequently through H; and therefore (35. and 36. iii.) the rectangle under G F, F H is equal to the rectangle under A F, F B. But as F H is equal to O E, the rectangle under G F, E O is equal to the rectangle under A F, F B; and as, by the eleventh Definition, the rectangle under A F, F B is equal to the square of C D, the rectangle under G F, E O is also equal to the square of C D.

Part II. Let A B be the transverse axis of the ellipse or hyperbola P B, of which C is the center, and let G E touch the section in the point P; the rectangle under the straight lines C K, M P, drawn from the transverse axis A B to the tangent G E, and at right angles to it, is equal to the square of C D, the semiconjugate axis.

Let the conjugate axis meet the tangent in the point I; and from the point P draw P N an ordinate to A B, and P L an ordinate to D C I. Then, as C K, M P are at right angles to the tangent G E, they are (28. i.) parallel to one another, and, by Cor. 2. Prop. IV. as P N is an ordinate to A B it is parallel to D C I, and as P L is an ordinate to D C I it is parallel to A B. Consequently (29. i.) the angle K C B is equal to the angle P M N; and as I C B, P N M are right angles, the angles I C K,

K C B

K C B together are equal to the angles M P N, P M N together, and therefore the angles I C K, M P N are equal, and the triangles I C K, M P N are equiangular. Hence C K : C I : : P N : P M. But (34. i.) P N is equal to C L, and therefore C K : C I : : C L : P M, and (16. vi.) C K × P M is equal to C I × C L. Confequently, as the rectangle under C I, C L, by Prop. VII. (and 17. vi.) is equal to the fquare of C D, the rectangle under C K, P M is alfo equal to the fquare of C D.

PROP. XVIII.

If a ftraight line touching an ellipfe or hyperbola be limited by tangents paffing through the vertices of the tranfverfe axis, the circumference of a circle defcribed about it as a diameter will pafs through the foci; and the rectangle under the two ftraight lines drawn from the point in which it touches the fection to the foci will be equal to the fquare of the femidiameter parallel to it.

Part I. Let the ftraight line G E touch the ellipfe or hyperbola P B in the point P, and meet in the points G, E the tangents A G, B E, paffing through the vertices A, B of the tranfverfe axis A B; the circumference of a circle defcribed about G E as a diameter will pafs through the foci F, O. Fig. 72. 73.

For, by the feventh Definition, and Cor. 2. Prop. IV. G A B, E B A are right angles; and, by the eleventh Definition, and Prop. IX. the rectangle under A G, B E is equal to the rectangle under A O, O B. Confequently (17. vi.) E B : B O : : A O : A G, and therefore the ftraight lines E O, G O being drawn, the angle E O B (6. vi.) is equal to the angle A G O, and the angle B E O is equal to the angle A O G. The angles E O B, A O G together are therefore equal to a right angle, and confequently (32. i.) the angle G O E is a

 right

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right angle. If therefore a circle be deſcribed about G E as a diameter, it is evident (from Prop. 31. iii. and 21. i.) that its circumference muſt paſs through the focus O; and in the ſame way it may be proved, that it muſt paſs through the focus F.

Part II. The reſt remaining as above, let P O, P F be drawn to the foci O, F, and let C D be the ſemidiameter parallel to G E; the rectangle under P F, P O is equal to the ſquare of C D.

For, draw O I perpendicular to G E, and, being produced, let it meet P F, or P F produced, in H. Then, by Cor. 1. to Prop. XV. the angles O P I, H P I are equal, and the angle O I P is equal to the angle H I P, each of them being a right angle, and P I is common to the two triangles O P I, H P I. Conſequently (26.i.) P O is equal to P H, and O I to I H; and it is evident (from 3. iii.) that the point H muſt be in the circumference of the circle deſcribed about G E as a diameter, and paſſing through F, O, according to Part I. The rectangle under F P, P H, or that under F P, P O, (35. and 36. iii.) is therefore equal to the rectangle under G P, P E. Conſequently, by Prop. IX. the rectangle under F P, P O is equal to the ſquare of C D.

PROP. XIX.

A ſtraight line drawn from either of the foci of an ellipſe or hyperbola, perpendicular to a tangent, is to a ſtraight line drawn from the ſame focus to the point of contact, as the ſemiconjugate axis to the ſemidiameter parallel to the tangent.

Fig. 70. 71.

Let P B be an ellipſe or hyperbola, of which the foci are F, O, and let G E touch the ſection in P, and let F G be perpendicular to it; the perpendicular F G is to the ſtraight line F P, joining the focus F and point of con-

contact, as C D the semiconjugate axis to C R the semidiameter parallel to G E.

For, draw O E perpendicular to G E, and draw P O. Then, by Cor. 1. to Prop. XV. the angles F P G, O P E are equal; and F G P, O E P being right angles, the triangles F P G, O P E, are equiangular. Consequently (4. vi.) F G : F P : : O E : O P, and by alternation F G : O E : : F P : O P; and therefore (22. vi.) $FG \times OE : FP \times OP :: FG^2 : FP^2$. But, by Prop. XVII. $FG \times OE$ is equal to CD^2, and $FP \times OP$ is equal to CR^2, by Prop. XVIII. Consequently $CD^2 : CR^2 :: FG^2 : FP^2$, and therefore (22. vi.) C D : C R : : F G : F P.

Cor. 1. The rest remaining as above, let C E parallel to F P meet the tangent G E in E, and let C K be perpendicular to G E and meet it in K. Then (4. vi.) F G : F P : : C K : C E. But, by Prop. XVI. C E is equal to C B, the semitransverse axis, and therefore, by the above, (and 11. v.) C K : C B : : C D : C R.

Cor. 2. The rest remaining as above, let the straight line P M, perpendicular to the tangent G E, meet the transverse axis A B in M, and then C B will be to C D as C R to P M. For, by the preceding Cor. C K : C B : : C D : C R, and therefore (1. vi.) $CK \times PM : CB \times PM :: CD^2 : CR \times CD$. But, by Prop. XVII. $CK \times PM$ is equal to CD^2, and therefore (14. v.) $CB \times PM$ is equal to $CR \times CD$. Consequently (16. vi.) C B : C D : : C R : P M.

PROP. XX.

If a straight line touch an ellipse or hyperbola, and from the point of contact two straight lines be drawn to an axis, the one an ordinate to it, the other perpendicular to the tangent, the segment of the axis between the center and ordinate will be to the segment between the perpendicular and ordinate as the axis to its parameter.

BOOK II.

Fig. 74. 75.

Let the ftraight line I R touch the ellip bola P B in the point P; and A B being th and D E the conjugate axis, and C the ce be an ordinate to A B, and P G an ordinate let the ftraight line M P be perpendicular meet A B in K and D E in M; then C H A B to its parameter, and C G is to G M a parameter.

Let the tangent I R meet A B in R, an Then, by Cor. 2. Prop. VII. the rectangle H R is equal to the rectangle under A I; R P K being a right angle, and P H being gles to K R, the rectangle under K H, H R the fquare of P H (Cor. 8. vi. and 16. vi.). C H : K H : : C H × H R : K H × H R; an on account of the equals, (and 11. v.) C A H × H B : P H^2. Confequently, as by A H × H B is to P H^2 as A B to its paramete K H (11. v.) as A B to its parameter.

Again (1. vi.) in the ellipfe, C G : G M : G M × G I; and as above, by Cor. 2. Pro rectangle under C G, G I is equal to the re der E G, G D, and (Cor. 8. vi.) the recta G M, G I is equal to the fquare of P G. C C G : G M : : E G × G D : P G^2; and th Prop. VI. (and 11. v.) C G is to G M as D parameter.

Laftly, in the hyperbola, as, by Cor. 2 P G is parallel to A B, and P H parallel to D G M as K P to P M, and therefore as K H to (1. vi.) K H : H C : : K H × R H : H C × (Cor. 8. vi. and 17. vi.) K H × R H is equa and, by Cor. 2. Prop. VII. H C × R H is eq × H B. Confequently (11. v.) C G : G M A H × H B. But, by Def. VIII. P H^2 is to

66.

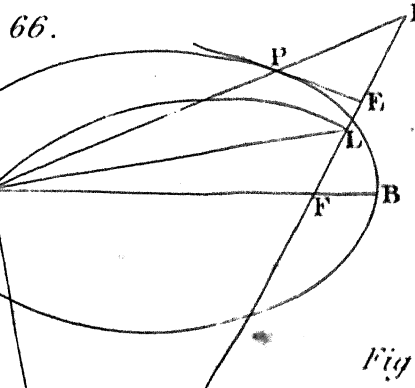


Fig. 68.

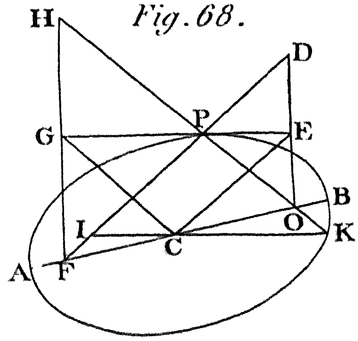


Fig. 70.

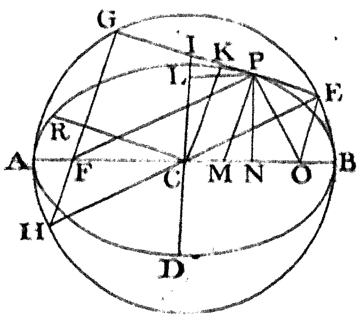


Fig. 69.

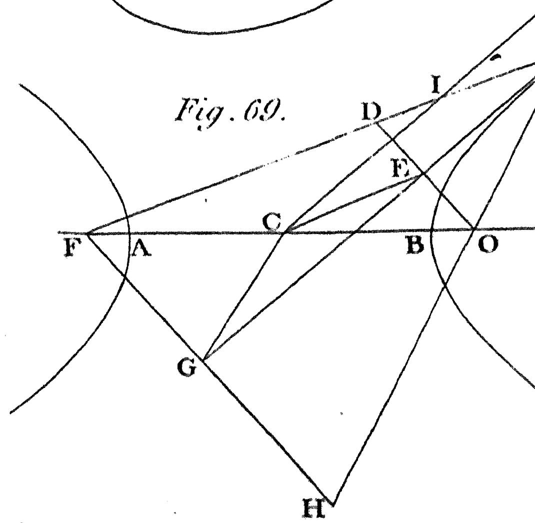


Fig. 67.

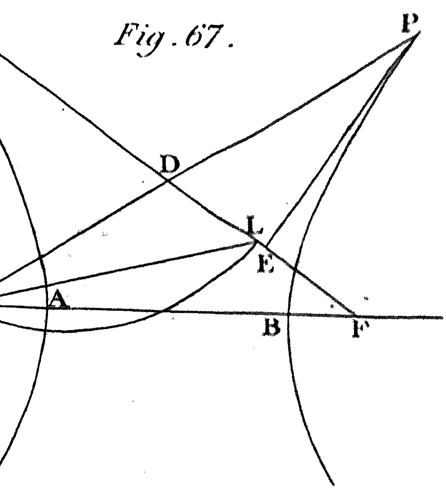
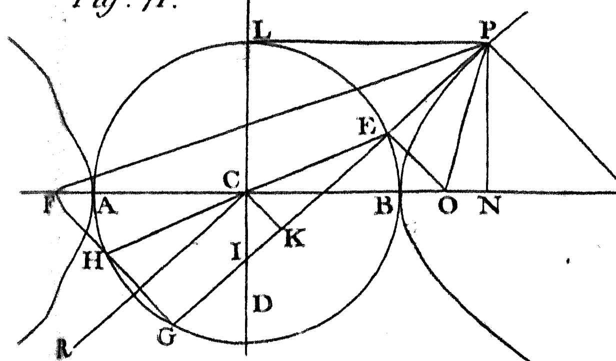


Fig. 71.



is evident from Def. IX. D E is to A B as D E is to its parameter. Consequently C G is to G M as D E is to its parameter.

PROP. XXI.

If a straight line touch an ellipse or hyperbola, and a straight line be drawn from the point of contact at right angles to it, and meet the axes, the rectangle under the segments of the perpendicular, between the point of contact and the axes, will be equal to the square of the semidiameter parallel to the tangent.

Let the straight line I R touch the ellipse or hyperbola P B in the point P, and let the straight line M P, at right angles to I R, meet the transverse axis A B in K, and the conjugate axis D E in M, and, C being the center, let C L be the semidiameter parallel to I R; the rectangle under P M, P K is equal to the square of C L. Fig. 74. 75.

For, by Cor. 2. Prop. XIX. and inversion, P K : C L : : C D : C B, and therefore $PK^2 : CL^2 :: CD^2 : CB^2$. And P G being drawn an ordinate to D E, by Prop. XX. C G is to G M as D E to its parameter. But, by Def. IX. (and Cor. 2. to 20. vi.) D E is to its parameter as DE^2 to AB^2, or (15. v.) as CD^2 to CB^2; and as P G is parallel to A B, (2. vi.) C G : G M : : P K : P M. Consequently (11. v.) $PK : PM :: CD^2 : CB^2$; and therefore (1. vi.) $PK^2 : PM \times PK :: CD^2 : CB^2$. But, by the above, $PK^2 : CL^2 :: CD^2 : CB^2$; and therefore (11. v.) $PK^2 : PM \times PK :: PK^2 : CL^2$. Consequently (14. v.) $PM \times PK$ is equal to CL^2.

Cor. By the above, and Prop. XVIII. the rectangle under P M, P K is equal to the rectangle under P O, P F, the straight lines drawn from P the point of contact to the foci O, F.

 PROP.

BOOK II.

PROP. XXII.

If a circle be described about the transverse axis of an ellipse as a diameter, a polygon may be inscribed in it, and a corresponding polygon in the ellipse, so that the polygon in the circle shall be to that in the ellipse as the transverse axis to the conjugate axis.

Fig. 76. Let A B be the transverse, and F G the conjugate axis of the ellipse A F B G, and about A B as a diameter let the circle A D B E be described; a polygon may be inscribed in A D B E, and a corresponding one in the ellipse A F B G, so that the polygon in the circle shall be to that in the ellipse as A B to F G.

For, let C be the common center of the ellipse and circle, and produce F G till it meet the circumference of the circle in D, E. Let K, M be points in the circumference, and draw D K, K M, M A. Draw K P, M O parallel to D C, and let them meet A C in P, O, and the curve of the ellipse in L, N, and draw F L, L N, N A. Draw K H, L I parallel to A B, and let them meet D C in H, I. Then P I, P H are parallelograms, and (34. i.) C H is equal to P K, and C I equal to P L; and K P, M O are perpendicular to A C, and L P, N O are ordinates to A B, by Cor. 2. Prop. IV. By Prop. V. $AC^2 : CF^2 :: AP \times PB : PL^2$, and therefore (35. iii.) as $AP \times PB$ is equal to KP^2, $AC^2 : CF^2 :: KP^2 : PL^2$, and (22. vi.) AC or $DC : CF :: KP$ or $CH : PL$ or CI*. Consequently (19. v.) $DC : FC :: DH : FI$; and $DH : FI :: CH : CI$. But (1. vi.) $CH : CI ::$ parallelogram PH : parallelogram PI; and $DH : FI ::$ the triangle DKH : the triangle FLI. Consequently

* This is the property referred to by writers on the Orthographical Projection of the Sphere, when they prove, that a circle, not parallel to the plane of projection, is projected into an ellipse.

(II.

(11. and 12. v.) D C : F C : : the trapezium D K P C : the trapezium F L P C. In the fame manner it may be demonftrated that D C : F C : : the trapezium K M O P : the trapezium L N O P; and alfo that D C : F C : : the triangle M A O : the triangle N A O. But (15. v.) D C : F C : : A B : F G, and therefore (12. v.) A B : F G : : the polygon D K M A C : the polygon F L N A C. Confequently, as infcriptions may be made in a fimilar manner all round the circle and ellipfe, the Prop. is evident.

Cor. A polygon may be infcribed in an ellipfe, which fhall be deficient from the ellipfe by a fuperficies lefs than any given fuperficies. For, the reft remaining as above, if a ftraight line parallel to L F be drawn to touch the ellipfe, and meet C F and P L produced, and from F, L ftraight lines be drawn to the point of contact, the triangle thus formed will be equal to half the parallelogram contained by L F, the tangent parallel to it, and C F, P L produced. This triangle therefore will be greater than half the elliptic fegment contained under the curve L F, and the ftraight line L F. Such an infcription therefore being made all round the ellipfe, the Cor is evident (from 1. x.).

PROP. XXIII.

If the tranfverfe axis of an ellipfe be alfo a diameter of a circle, the ellipfe will be to the circle as the conjugate axis to the tranfverfe axis.

Let A B be the tranfverfe axis of the ellipfe A F B G, and alfo a diameter of the circle A D B E; the ellipfe is to the circle as F G the conjugate axis to A B the tranfverfe axis.

For, every thing remaining as in Prop. XXII. let the circle Q R S be to the circle A D B E as F G to A B; and then, if it can be proved that the circle Q R S is equal Fig. 76.

BOOK II.

equal to the ellipse A F B G, the truth of the Proposition will be manifest. If the circle Q R S be not equal to the ellipse, let it first, if possible, be greater. Then it is possible to inscribe in the circle Q R S a polygon, having an even number of sides, and greater than the ellipse A F B G. Let it be understood to be inscribed, and let a polygon similar to it be supposed to be inscribed in the circle A D B E; and from the angular points of the polygon in A D B E let straight lines be drawn parallel to D E. Let the points in which these parallel lines cut the curve of the ellipse be joined, and then a polygon will be inscribed in the ellipse corresponding to the polygon in the circle A D B E, as in the last Proposition; and A B : F G : : the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse. But, by hypothesis and inversion, the circle A D B E : the circle Q R S : : A B : F G; and therefore (11. v.) the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse : : the circle A D B E : the circle Q R S. Consequently (1. and 2. xii.) the polygon inscribed in the circle A D B E : the polygon inscribed in the ellipse : : the polygon inscribed in the circle A D B E : the polygon inscribed in the circle Q R S. The polygon (14. v.) inscribed in the ellipse is therefore equal to the polygon inscribed in the circle Q R S: which is absurd; for, by the present hypothesis, the polygon inscribed in the circle Q R S is greater than the ellipse.

Secondly, if it be possible, let the circle Q R S be less than the ellipse. Then it is possible, by Cor. Prop. XXII. to inscribe in the ellipse a polygon greater than the circle Q R S, and a polygon corresponding to it in the circle A D B E; and to inscribe in the circle Q R S a polygon similar to the polygon inscribed in the circle A D B E. Let such polygons be supposed to be so inscribed.

lygon inscribed in the circle Q R S, contrary to the construction which has now been supposed to be made. The circle Q R S therefore is equal to the ellipse A F B G, and therefore the ellipse A F B G is to the circle A D B E as F G to A B.

Cor. 1. An ellipse is equal to a circle, whose diameter is a mean proportional between its axes. For, by the above, F G : A B : : the ellipse A F B G, or the circle Q R S : the circle A D B E. But (1. vi.) F G : A B : : F G $\times$ A B : A B^2; and (2. xii.) the circle Q R S : the circle A D B E : : Q S^2 : A B^2, Q S being the diameter of the circle Q R S. Consequently (11. v.) the ellipse A F B G : circle A D B E : : Q S^2 : A B^2 : : F G $\times$ A B : A B^2 : and (14. v.) Q S^2 is equal to F G $\times$ A B.

Cor. 2. From the preceding (and 17. vi. and 2. xii.) it is evident, that the areas of two ellipses are to one another as the rectangles under their axes.

The Cor. to Prop. XIV. Lib. I. of the Principia depends, in a great degree, upon this truth.

PROP. XXIV.

If from a point in the conjugate axis of an ellipse a straight line, equal to the difference of the semiaxes, be drawn to a point in the transverse axis, and be produced beyond the transverse axis, so that the part produced be equal to the semiconjugate axis, the extremity of the part produced will be in the curve of the section. Or, if from a point in the conjugate axis of an ellipse a straight line, equal to the sum of the semiaxes, be drawn to a point in the transverse axis, and if this line be so cut that the segment between the transverse axis and the point

BOOK II.

point of ſection be equal to the ſemiconjugate axis, the point of ſection will be in the curve.

Fig. 77. 78. Let A D B E be an ellipſe, of which A B is the tranſverſe, and D E is the conjugate axis, and C is the center. Let F be a point in the conjugate, and G be a point in the tranſverſe axis, and let the ſtraight line F G be equal to the difference or ſum of C B, C D. When F G is equal to the difference of C B, C D, as in Fig. 78. let F G be produced beyond A B to H, ſo that G H be equal to C D; but when F G is equal to the ſum of C B, C D, as in Fig. 77. let the ſegment G H be equal to C D; and in either caſe the point H will be in the curve of the ellipſe.

For through the center C draw the ſtraight line C K parallel to F H. Through H draw the ſtraight line H K parallel to D E, and let it meet C K in K, and A B in I. Then (34. i.) the ſtraight line C K is equal to F H. But as F G is equal to the ſum or difference of C B, C D, and as G H is equal to C D, the ſtraight line F H is equal to C B; and conſequently C K is equal to C B. With C, therefore, as a center, and C B as a diſtance, let a circle be deſcribed, and it will paſs through K. Again, on account of the ſimilar triangles C K I, G H I, $CK^2 : GH^2 :: KI^2 : HI^2$; and therefore, on account of the equals, $CB^2 : CD^2 :: KI^2 : HI^2$. But (3. and 35. iii.) the ſquare of K I is equal to the rectangle under A I, I B; and therefore $CB^2 : CD^2 :: AI \times IB : HI^2$. The point H is therefore in the curve, by Cor. 1. Prop. V. (and 9. v.) for H I is parallel to the ordinates of A B.

SCHOLIUM.

The inſtrument called by ſome *the trammels*, and by others

others the *elliptic compaſſes*, uſed by cabinet-makers, &c. for deſcribing the curves of ellipſes, are conſtructed on the property demonſtrated in this Propoſition. As the trammels are in general uſe, it is needleſs to give a deſcription of them in this place. Lathes for making picture-frames, and ornaments of an elliptical form, are conſtructed on the ſame property.

A GEO-

A

GEOMETRICAL TREATISE

OF

CONIC SECTIONS.

BOOK III.

Of the Parabola, the Directrices of the Sections, the Asymptotes of the Hyperbola, Conjugate Hyperbolas, and of hyperbolic Sectors and Trapezia.

DEFINITIONS.

I.

THE section F D C being a parabola, and V B E its vertical plane, as in the 15th and 16th Definitions in the first Book, any straight line, as D I, in the parabola parallel to V B, the side in which V B E touches the cone, is called a *Diameter* of the parabola. Fig. 79.

Cor. 1. From this Definition (and 9. xi.) the diameters of a parabola are parallel to one another; and, by Prop. XIV. Book I. any straight line drawn in the plane of a parabola, parallel to a diameter, will meet the curve in one point, and in one point only, and, by this Definition, it will itself be a diameter.

Cor.

BOOK III.

Cor. 2. Any ſtraight line in a parabola, not parallel to a diameter, will meet the curve in two points. For any ſtraight line drawn through V, the vertex of the cone, and in the vertical plane, and not in the ſame direction with V B, will fall without the oppoſite cones; and by the demonſtration of the ſecond part of Prop. VIII. of the firſt Book, one plane may be drawn through this ſtraight line to touch the conical ſuperficies, and cut the plane of the parabola. The interſection alſo of this plane with the plane of the parabola will touch the parabola, and any ſtraight line in the parabola parallel to this tangent will meet the curve in two points, by Cor. 1. Prop. VIII. Book I. Hence (16. xi.) the Cor. is evident.

II.

The point in which a diameter of a parabola meets the curve is called the *Vertex* of the diameter.

III.

If a ſtraight line terminated by the curve of a parabola be biſected by a diameter, it is called a *Double Ordinate* to that diameter; and its half is ſimply called an *Ordinate* to it.

IV.

The ſegment of a diameter between its vertex and an ordinate is called an *Abſciſs* of that diameter.

V.

The diameter of a parabola, which cuts its ordinates at right angles, is called the *Axis* of the parabola.

VI.

A third proportional to an abſciſs of a diameter of a parabola, and the correſponding ordinate, is called the *Parameter*, or *Latus Rectum* of the diameter. The parameter of the axis is frequently called the *Principal Parameter*, or *Latus Rectum*.

PROP.

BOOK III.

PROP. I.

If each of two diameters of a parabola meet a ſtraight line, and if each of theſe ſtraight lines cut, or one of them cut and the other touch the parabola, and if theſe two ſtraight lines be parallel; then the ſegment of the one diameter, between its vertex and the line which it meets, will be to the ſegment of the other between its vertex and the line which it meets, as the ſquare of the line which meets the firſt mentioned diameter if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of the line which meets the other diameter if a tangent, or the rectangle under its ſegments if a ſecant.

Suppoſe D I, G K to be two diameters of a parabola, and let D be the vertex of the one, and G the vertex of the other. Let L P, M N be two parallel ſtraight lines, and let D I meet L P in L, and G K meet M N in M, and let L P, M N either both cut, or one of them cut and the other touch the parabola; then D L is to G M as the ſquare of L P, if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N, if a tangent, or the rectangle under its ſegments if a ſecant. Fig. 80.

For let F G D C be the parabola as formed in the cone, and D I, G K the diameters mentioned above. Let the parabola cut the plane of the baſe in the ſtraight line F K I C, and let the vertical plane cut it in B E, V B being the ſide along which the vertical plane touches the cone. Then D I, G K are parallel to V B, by the firſt Definition. Through the parallels V B, D I let a plane paſs, and let it cut the cone in the ſide V D A, and the baſe in B I A. Through the parallels V B, G K let a plane paſs, and let it cut the cone in the ſide V G H, and the baſe in B K H. Then (4. vi. and 16. v.) Fig. 79.

D I : V B :: A I : A B, and

V B : G K :: H B : H K.

H Hence

BOOK III. Hence (1. vi.) DI : VB :: AI × IB : AB × IB and VB : GK :: HB × KB : HK × KB. But (35. iii.) AI × IB is equal to FI × IC, and HK × KB is equal to FK × KC; and, by the seventh Lemma, AB × IB is equal to HB × KB. Consequently, by the above and substitution, we have the two following ranks of magnitudes proportionals, taken two and two in the same order,

$$\begin{array}{ccccc} DI & : & VB & : & GK \\ FI \times IC & : & AB \times IB & : & FK \times KC \end{array}$$

and therefore (22. v.) DI : GK :: FI × IC : FK × KC. Let the straight line FKIC have the same situation in the parabola in Fig. 80. as in Fig. 79. and first suppose LP, MN to be parallel to the base of the cone, or to FC. Then, by Prop. XVI. Book I. DL is to DI as the square of LP, if a tangent, or the rectangle under its segments, if a secant, to FI × IC; and, by the above, DI is to GK as FI × IC to FK × KC; and again, by Prop. XVI. Book I. GK is to GM as FK × KC to the square of MN if a tangent, or the rectangle under its segments if a secant. We have therefore

$$\begin{array}{ccccccc} DL & : & DI & : & GK & : & GM \\ \left.\begin{array}{c} t.\ LP^2 \\ \text{or} \\ f.\ LP^r \end{array}\right\} & : & FI \times IC & : & FK \times KC & : & \left\{\begin{array}{c} t.\ MN^2 \\ \text{or} \\ f.\ MN^r \end{array}\right. \end{array}$$

Consequently (22. v.) DL is to GM as the square of LP if a tangent, or the rectangle under its segments if a secant, to the square of MN if a tangent, or the rectangle under its segments if a secant. Lastly, let LP, MN not be parallel to the base of the cone, but let LS, MR be parallel to the base, and let them touch or cut either of the conical superficies. Then, by the above, DL is to GM as the square of LS if a tangent, or the rectangle under its segments if a secant, to the square of MR if a tangent, or the rectangle under its segments if

if a ſecant. But, by Prop. XII. Book I. the ſquare of L S if a tangent, or the rectangle under its ſegments if a ſecant, is to the ſquare of M R if a tangent, or the rectangle under its ſegments if a ſecant, as the ſquare of L P if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N if a tangent, or the rectangle under its ſegments if a ſecant. Conſequently (11. v.) D L is to G M as the ſquare of L P if a tangent, or the rectangle under its ſegments if a ſecant, to the ſquare of M N if a tangent, or the rectangle under its ſegments if a ſecant.

PROP. II.

A diameter of a conic ſection biſects any ſtraight line it meets in the ſection parallel to a tangent paſſing through its vertex; and ordinates to a diameter, and a tangent paſſing through its vertex, are parallel to one another.

In the ellipſe and hyperbola this has been proved, according to Cor. 3. to Prop. III. Book II. In the parabola A B C let the diameter B G cut the ſtraight line A C in the point G, and let A C be parallel to D E touching the parabola in B, the vertex of the diameter B G; the ſtraight line A C is biſected in G. On the contrary, any ſtraight line in the parabola biſected by B G is parallel to A C, or the tangent D E. Fig. 81.

Part I. Through A, C let A D, C E be drawn parallel to B G, and let A D meet the tangent in D, and C E meet it in E. Then, by Cor. 1. Def. I. A D, C E are diameters, and, by Prop. I. A D : C E : : $DB^2 : BE^2$; and (34. i.) as A D, C E are equal, it follows that D B, B E are equal to one another. Conſequently (34. i.) A G is equal to G C.

Part II. If it be poſſible, let the ſtraight line H R in the parabola A B C be biſected by the diameter B G, and not be parallel to A C or D E.

Through

BOOK III.

Through R draw R M parallel to the diameter B G, and through H draw H L parallel to A C or D E, and let it meet B G in K, the curve again in L, and R M in M. Let B G meet H R in N. Then as H R is biſected in N, and as K N, M R are parallel, (2. vi.) H N : N R : : H K : K M, and H K is equal to K M. But, by Part I. H K is equal to K L, and therefore K L is equal to K M; which is abſurd. Conſequently no ſtraight line in the parabola, unleſs it be parallel to D E, or to the ordinate A C, can be biſected by the diameter B G. Ordinates to the diameter B G muſt therefore be parallel to one another, and to the tangent D E, paſſing through the vertex.

Cor. 1. From hence, and Prop. III. Book II. it is evident, that if a ſtraight line be an ordinate to a diameter, any ſtraight line in the ſection, or oppoſite ſection, and parallel to it, will be an ordinate to the ſame diameter.

Cor. 2. From this Propoſition it is evident, that if a ſtraight line biſect two parallel lines in a conic ſection, it will be a diameter.

Cor. 3. From the above a method of finding a diameter of a given parabola is evident. For two parallel ſtraight lines being drawn in the parabola, a ſtraight line biſecting them, and any ſtraight line parallel to it, will be a diameter.

Cor. 4. The method of finding the axis of a parabola is alſo evident from the above. For, having found a diameter of the parabola, by the preceding Cor. let a ſtraight line at right angles to it be drawn within the parabola, and limited both ways by the curve. Then the diameter biſecting this ſtraight line will be the axis; for, being parallel to the diameter firſt found, it will (29. i.) biſect the ſtraight line in the ſection at right angles.

PROP.

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81.

Fig. 72.

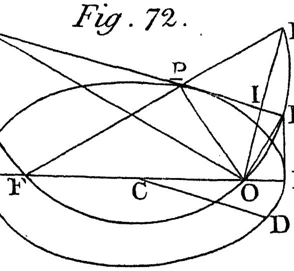
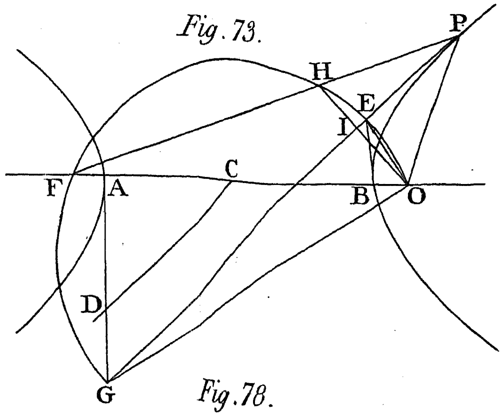


Fig. 73.



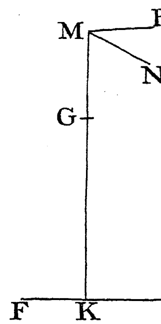


Fig. 77.

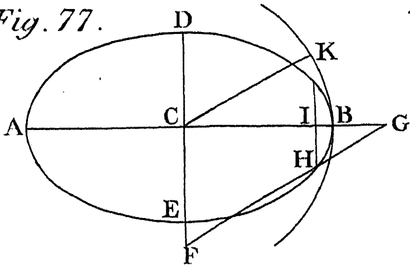
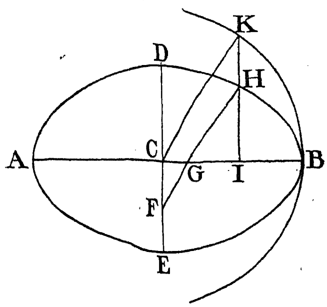


Fig. 78.



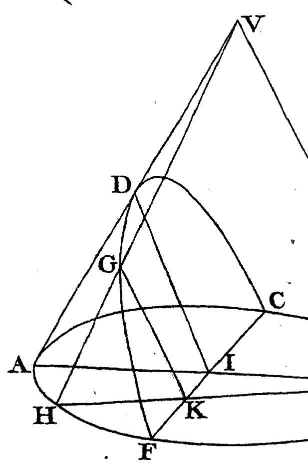


Fig. 74.

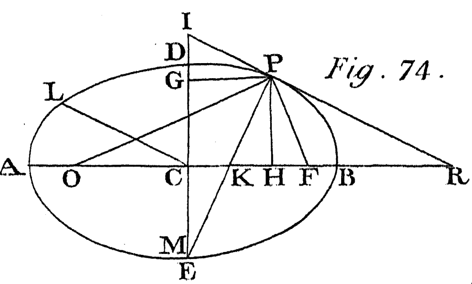


Fig. 75.

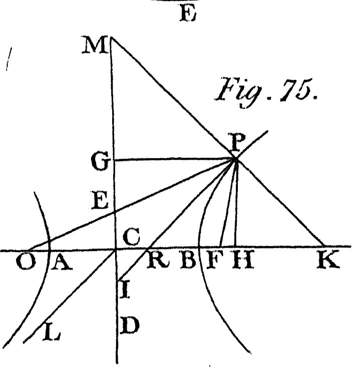
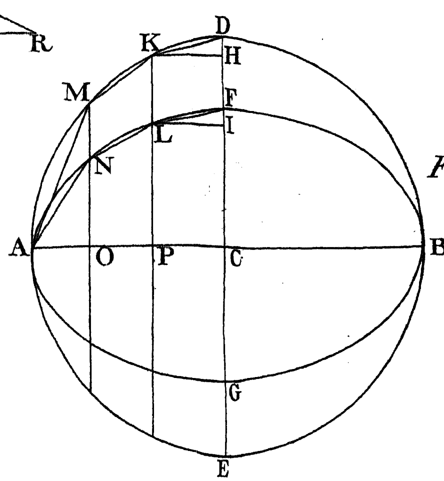
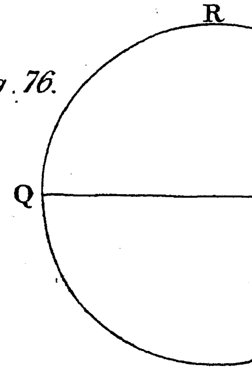


Fig. 76.





BO
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PROP. III.

BOOK III.

The abscisses of a diameter of a parabola are to one another as the squares of the corresponding ordinates; and the square of an ordinate to a diameter of a parabola is equal to the rectangle under the parameter of the diameter, and the abscis corresponding to the ordinate.

Part I. Let B G be a diameter of the parabola A B C, and let A G, H K be ordinates to it, and let them meet it in the points G, K, and let B be the vertex of the diameter; the abscis B G is to the abscis B K as the square of the ordinate A G to the square of the ordinate H K. Fig. 81.

For, as B G is parallel to a side of the cone, in which the section was formed, and as A G, H K, by Prop. II. are parallel, and as they would be bisected in G, K if limited by the curve, this part is evident from Prop. XVI. Book I.*

Part II. The rest remaining as above, let the straight line P be a third proportional to the abscis B G and the corresponding ordinate A G, and consequently the parameter to the diameter B G, according to the sixth Definition; the square of the ordinate H K is equal to the rectangle under P and the abscis B K.

For, by the preceding part, $BG : BK :: AG^2 : HK^2$, and therefore (1. vi.) $P \times BG : P \times BK :: AG^2 : HK^2$. But (17. vi.) $P \times BG$ is equal to AG^2, and consequently (14. v.) $P \times BK$ is equal to HK^2.

Cor. 1. If a straight line touching a parabola meet a diameter, the square of the segment, between the point of contact and the point of concourse, will be equal to

* From this and the second part of Prop. II. writers on projectiles prove, that, if the resistance of the air have no perceptible effect, a projectile must move in the curve of a parabola.

BOOK III.

the rectangle under the ſegment of the diameter, between its vertex and the point of concourſe, and the parameter of the diameter, to whoſe ordinates the tangent is parallel. For let D B, touching the parabola in B, meet the diameter A D in D, and let A G parallel to D B meet the diameter B G in G, and let P be the parameter of B G. Then (34. i.) A G is equal to D B, and A D is equal to B G, and, by Prop. II. D B is parallel to the ordinates of B G; and as, by the above, B G $\times$ P is equal to A G^2, B D^2 is equal to A D $\times$ P.

Cor. 2. If a ſtraight line cutting a parabola meet a diameter, the rectangle under its ſegments, between the point of concourſe and the curve, will be equal to the rectangle under the ſegment of the diameter between its vertex and the point of concourſe, and the parameter of the diameter, to whoſe ordinates the ſecant is parallel. For, the reſt remaining as in the preceding Cor. let the tangent B D be parallel to the ſtraight line H M, cutting the parabola in H, L, and meeting the diameter R M in M. Then, by Prop. I. A D : R M : : D B^2 : H M $\times$ M L; and therefore (1. vi.) A D $\times$ P : R M $\times$ P : : D B^2 : H M $\times$ M L. Conſequently, by the preceding Cor. (and 14. v.) R M $\times$ P is equal to H M $\times$ M L.

SCHOLIUM.

On account of the equality of the ſquare of H K to the rectangle under P and B K, Apollonius called the ſection a parabola.

From the property demonſtrated above the parabola is frequently denoted by an algebraical equation, in the following manner. Put the parameter of the diameter B K $= p$, the abſciſs B K $= x$, and the ordinate H K $= y$. Then, from the above, $x : y : : y : p$, and $p\,x = y^2$.

PROP.

PROP. IV.

If each of two ſtraight lines meeting one another touch or cut, or one of them touch and the other cut, a parabola, the ſquare of the firſt of the two if a tangent, or the rectangle under its ſegments if a ſecant, will be to the ſquare of the ſecond if a tangent, or the rectangle under its ſegments if a ſecant, as the parameter of the diameter to whoſe ordinates the firſt is parallel, to the parameter of the diameter to whoſe ordinates the ſecond is parallel.

For, firſt let the ſtraight lines A E, C E, meeting one another in E, touch the parabola in A and C, and let E B be a diameter paſſing through E. Then, by Cor. 1. Prop. III. the ſquare of A E is equal to the rectangle under B E, and the parameter of the diameter to whoſe ordinates A E is parallel; and the ſquare of C E is equal to the rectangle under B E, and the parameter of the diameter to whoſe ordinates C E is parallel. Conſequently (1. vi.) the ſquare of A E is to the ſquare of C E, as the parameter of the diameter to whoſe ordinates A E is parallel to the parameter of the diameter to whoſe ordinates C E is parallel. Fig. 82.

Next, let the ſtraight line G A, touching the parabola in A, meet in G the ſtraight line G K, which cuts the parabola in F, K, and let G D be a diameter paſſing through G. Then, by Cor. 1. Prop. III. the ſquare of A G is equal to the rectangle under D G, and the parameter of the diameter to whoſe ordinates A G is parallel; and, by Cor. 2. Prop. III. the rectangle K G F is equal to the rectangle under D G, and the parameter of the diameter to whoſe ordinates G K is parallel. Conſequently (1. vi.) the ſquare of A G is to the rectangle K G F as the parameter of the diameter to whoſe ordi-

BOOK III. nates A G is parallel, to the parameter of the diameter to whose ordinates G K is parallel.

Lastly, if the straight line G K, cutting the parabola in F, K, meet in the point G the straight line G L, which cuts the parabola in the points H, L, then it may be proved in the same way, by means of Cor. 2. Prop. III. that the rectangle K G F is to the rectangle L G H as the parameter of the diameter to whose ordinates G K is parallel, to the parameter of the diameter to whose ordinates G L is parallel.

PROP. V.

If a straight line touching a parabola meet a diameter, and an ordinate to the diameter pass through the point of contact, the segment of the diameter, between its vertex and the tangent, will be equal to its abscis, between its vertex and the ordinate.

Fig. 83. Let the straight line A E, touching the parabola A B C in the point A, meet the diameter B D in the point E, and through the point of contact A let the ordinate A D to B D pass, and meet B D in D; the segment B E between B the vertex and the tangent is equal to the abscis B D between the vertex and the ordinate.

For produce A D till it meet the curve in C, and draw C F parallel to E D, and let it meet A E in F. Then, by the third Definition, A C is bisected in D, and, by Cor. 1. to the first Definition, C F is a diameter; and, by Prop. I. $BE : CF :: AE^2 : AF^2$. But C F, D E being parallel, (2. vi.) $AD : DC :: AE : EF$, and A C being bisected in D, A F is bisected in E, and for the same reasons D E is half of C F. Consequently AF^2 is equal to four times AE^2 (4. ii.) and, therefore, C F is equal to four times B E; and as C F is double of D E, B E is equal to B D.

Cor.

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Cor. If A G, B D be any two diameters of the parabola A B C, and if A D be an ordinate to B D, and B G be an ordinate to A G, the abſciſſes A G, B D will be equal. For let A E touch the parabola in A, and meet the diameter B D in E. Then, by Cor. 1. to the firſt Definition, A G, E B are parallel, and, by Prop. II. A E, G B are parallel. Conſequently (34. i.) A G is equal to E B, and therefore, as by the above E B, B D are equal, A G is equal to B D.

PROP. VI.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, the diameter biſecting the line joining the points of contact will paſs through the point of concourſe.

Fig. 83.

In the ellipſe, hyperbola, or oppoſite hyperbolas, this has been proved in Prop. VIII. Book II. In the parabola A B C let the two ſtraight lines E A, E C touch the ſection in the points A C, and meet one another in E, and let the diameter B D biſect A C, the ſtraight line joining the points of contact in D; the diameter B D will paſs through E.

For, as A C is biſected by the diameter B D, it is a double ordinate to B D; and therefore, by Prop. V. if B D be produced and meet the tangents, its ſegment between B the vertex and the tangent A E will be equal to its abſciſs B D; and its ſegment between B and the tangent C E will alſo be equal to B D. The diameter B D will therefore meet both the tangents A E, C E in the ſame point, and conſequently will paſs through E, the point of concourſe.

Cor. 1. If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, a ſtraight line paſſing through the point of concourſe, and biſecting

BOOK III.

ing the line joining the points of contact, will be a diameter. The truth of this is evident from Cor. Prop. VIII. Book II. and the above.

Cor. 2. From the above it is evident, that if A C be a double ordinate to B D, a diameter of any conic section A B C, and if A E, touching the section in A, meet the diameter in E, then if the straight line E C be drawn, it will touch the section in C.

DEFINITIONS.

VII.

Fig. 84. If from B, the vertex of the axis A B of the parabola P B M, a segment B F be taken in the axis equal to one fourth of the parameter of the axis, the point F is called the *Focus*, or *Umbilicus*, of the parabola.

Cor. The double ordinate T S, drawn through F the focus of any conic section, is equal to the parameter of the axis passing through the focus. This has been proved in the ellipse and hyperbola in Cor. 4. to the eleventh Definition in Book II. In the parabola the square of T F is equal to the rectangle under B F, and four times B F, by this Def. and Prop. III. and therefore 4 TF^2 is equal to 4 B F × 4 B F. But (4. ii.) 4 TF^2 is equal to TS^2, and consequently T S is equal to 4 B F, and therefore, by this Def. equal to the parameter of the axis A B.

VIII.

As in the ellipse and hyperbola, so in the parabola, the straight line touching the section in T, the extremity of the double ordinate drawn as above, is called the *Focal Tangent* to the parabola.

IX.

The transverse axis of an ellipse or hyperbola, and the axis of a parabola, is sometimes called the *Focal Axis* of the section.

X. If

X.

BOOK III.

If the focal tangent T G, belonging to the focus F in any conic ſection P B M, meet the focal axis A B in X, the ſtraight line X Y at right angles to A B is called a *Directrix* of the ſection. And, if in the ellipſe or hyperbola O be the other focus, and the focal tangent belonging to O meet the focal axis A B in K, the ſtraight line K L at right angles to A B is alſo called a directrix of the ellipſe or hyperbola. Fig. 84. 85. 86.

Cor. 1. As in the ellipſe and hyperbola the foci F, O are equally diſtant from C the center, it is evident from the above, and Prop. VII. Book II. that the directrices X Y, K L are equally diſtant from the center.

Cor. 2. In the parabola, the focus F and the directrix X Y are equally diſtant from B, the vertex of the axis, by the above and Prop. V.

PROP. VII.

If a tangent paſſing through the vertex of the focal axis of a conic ſection meet a focal tangent, its ſegment between the point of contact and point of concourſe will be equal to the ſegment of the axis between the point of contact and the focus to which the focal tangent belongs.

Let the tangent B G, paſſing through B the vertex of the focal axis A B, of any conic ſection P B M, meet in the point G the focal tangent T G belonging to the focus F; the ſegment G B is equal to the ſegment F B. And in the ellipſe and hyperbola the tangent A H, paſſing through A the other vertex of the focal axis, and meeting the focal tangent T G in H, is equal to the ſegment A F. Fig. 84. 85. 86.

As far as this Propoſition relates to the ellipſe, or hyperbola, it has been proved in Prop. XII. Book II.

In

BOOK III.

In the parabola, every thing remaining as in the feventh Definition and its Cor. T F is double of F B, and therefore, by Cor. 2. to the tenth Definition, T F is equal to F X. Again, by Prop. II. T F, G B are parallel, and therefore (4. vi.) T F : F X : : G B : B X, and G B is equal to B X, and confequently equal to F B.

PROP. VIII.

A ftraight line drawn from any point in the curve of a conic fection to a focus is to a ftraight line drawn from the fame point perpendicular to the directrix neareft this focus, as the fegment of the axis between the fame focus and the neareft vertex, to the fegment between this vertex and the directrix: and, in the ellipfe and hyperbola, a ftraight line drawn from the fame point in the curve to the other focus is to a ftraight line drawn from the fame point perpendicular to the other directrix in the fame ratio.

Fig. 84. 85. 86.

A ftraight line P F, drawn from any point P in the curve of the conic fection P B M to the focus F, is to P Y perpendicular to X Y, the directrix neareft to F, as the fegment F B, of the focal axis between F and the vertex B, to the fegment B X of the fame axis between B the vertex and the directrix: and, in the ellipfe and hyperbola, P O drawn to the other focus O is to P Q drawn perpendicular to the other directrix K L in the fame ratio.

For, the reft remaining as in the preceding Prop. and Def. X. through P draw P M an ordinate to the axis A B, and let it meet the curve again in M, the axis A B in R, and the focal tangent T G in N. Then, by Prop. XIII. Book I. $TG^2 : TN^2 :: GB^2 : PN \times NM$. But, by Prop. II. N M, T F, G B are parallel, and therefore (10. and 22. vi.) $TG^2 : TN^2 :: FB^2 : FR^2$; and therefore

(11.

(11. v.) $FB^2 : FR^2 :: GB^2 : PN \times NM$. Consequently, as by Prop. VII. FB is equal to GB, FB^2 is equal to GB^2, and (14. v.) FR^2 is equal to $PN \times NM$; and therefore (6. ii. and 47. i.) RN^2 is equal to PF^2, and RN is equal to PF. But (4. vi.) RN : RX :: GB : BX, and therefore as (34. i.) PY is equal to RX, and, by Prop. VII. FB is equal to GB, PF : PY :: FB : BX.

Again, in the ellipse and hyperbola, the rest remaining as above, let C be the center, and let NV perpendicular to the directrix KL meet KL in V. Then (34. i.) NV, PQ are equal, and VK is equal to NR, and consequently equal to PF. Let CD the semiconjugate axis be produced till it meet the focal tangent TG in I, and, by Cor. 2. Prop. XIII. Book II. CI will be equal to CB or CA. Also (4. vi.) XC : CI :: XK : KL; and therefore, as XC, CK are equal, KL is equal to AB the transverse axis. Consequently, by Cor. 1. Prop. XIII. Book II. LV is equal to PO; and as LV is parallel to GB, and VN to BX, LV : VN :: GB : BX. On account of the equals therefore, PO : PQ :: FB : BX.

Cor. 1. In the ellipse and hyperbola, (4. vi.) CI : CX :: NR : RX; and therefore on account of the equals CB : CX :: PF : PY. But, by Prop. VII. Book II. CB : CX :: CF : CB; and therefore (11. v.) CF : CB :: PF : PY. For the same reasons, or by the above, (and 11. v.) CF : CB :: FB : BX.

Cor. 2. By the preceding Cor. in the ellipse PF is less than PY, but in the hyperbola PF is greater than PY. And, as by the above (and 11. v.) PF : PY :: PO : PQ, in the ellipse PO is less than PQ, but in the hyperbola PO is greater than PQ.

Cor. 3. A straight line drawn from any point in the curve of a parabola to the focus is equal to a straight line

Fig. 84.

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line drawn from the fame point perpendicular to the directrix. For P F : P Y : : F B : B X, and F B is equal to B X.

SCHOLIUM.

Some writers on conic fections have chofen this property as the primary one for their treatifes, and according to it have defined the fections in the following manner.

Fig. 84. 85. 86. Let F be a point without the ftraight line X Y, and whilft a ftraight line F P revolves about F as a center, let a point P fo move in F P that F P may always be to P Y, perpendicular to X Y, in a given ratio. The curve defcribed by the point P will be a conic fection; and it will be a parabola, ellipfe, or hyperbola, according as F P is equal to, lefs, or greater than P Y.

PROP. IX.

If from any point in the curve of a parabola a ftraight line be drawn to the focus, and a ftraight line perpendicular to the directrix, the angle contained by thefe ftraight lines will be bifected by a tangent paffing through the fame point.

Fig. 89. From the point P in the curve of the parabola P B R let the ftraight line P F be drawn to F the focus, and the ftraight line P D perpendicular to D X the directrix; the ftraight line P E, touching the parabola in P, bifects the angle F P D.

For let the tangent P E meet the axis A B in E, and let P A be an ordinate to A B. Then, by Prop. V. A B is equal to B E; and as, by Cor. 2. to the tenth Definition, F B is equal to B X, A X is therefore equal to F E. But (34. i.) A X is equal to P D; and, by Cor. 2. Prop. VIII. P D is equal to P F. Confequently P F,

F E

F E are equal, and therefore (5. i.) the angle F P E is equal to the angle F E P. But as P D, A E are parallel, the angle F E P (29. i.) is equal to the angle D P E. Confequently the angle F P E is equal to the angle D P E, and therefore the angle F P D is bifected by the tangent P E.

Cor. If a ftraight line touching a parabola meet the axis, the fegment of the axis between the point of concourfe and the focus is equal to the ftraight line drawn from the point of contact to the focus. This is evident from the above, for F E is equal to P F.

SCHOLIUM.

It is not certain when, or by whom, the name *Focus*, or *Umbilicus*, was firft given to a point in a conic fection. Neither of the two occurs in the Treatife of Apollonius, or in the writings of Archimedes, who occafionally mentions properties of the fections. The points themfelves, however, in the ellipfe and hyperbola, were well known to Apollonius. He calls them *puncta ex applicatione facta*; and he demonftrates the moft important properties of lines related to them. He does not mention the focus of the parabola.

It is highly probable, from analogy of general principles, and from the hiftory of this branch of fcience as far as it has been traced, that optical purfuits firft fuggefted each of the two names. For, in opticks, if a ray of light fall upon a plane furface, the angle of incidence is equal to the angle of reflexion; and if a ray of light fall upon a curve, and a ftraight line touch the curve in the fame point, the angle contained by the incident ray and tangent will be equal to the angle contained by the reflected ray and tangent, as a tangent is the direction of a curve in the point of contact.

These

BOOK III.

Fig. 89. 90. 91.

Thefe truths being premifed, fuppofe a conic fection R B P to revolve about A B, the focal axis, and that a concave fpeculum is formed by the curve by this revolution. Let the ftraight line T P E touch any one of the fections in the point P. In the ellipfe and hyperbola let F, O be the foci, and let F be the focus of the parabola.

Fig. 91.

1. Let K P be a ray of light, whofe direction, in a ftraight line, paffes through O the focus oppofite to F, within the hyperbolic fpeculum R B P. Let it fall upon the fpeculum in the point P, and draw P F, and produce K P to O. Then, as by Prop. XV. Book II. the angle F P E is equal to the angle O P E, and confequently (15. i.) equal to the angle K P T, the ftraight line P F will reprefent the ray of light after reflexion. For, if a perpendicular to T E be drawn from P, the angle contained by it and K P will be equal to the angle contained by it and F P; the angle of incidence being equal to the angle of reflexion. Hence if any number of rays fall on a concave hyperbolic fpeculum, and be converging to the focus in the oppofite hyperbola, they will be reflected to the focus within the fpeculum.

Fig. 90.

2. Let A R D B P be a concave elliptic fpeculum. Let a ray of light proceeding from the focus O fall upon the fpeculum at P, and let P F be drawn. Then will P F reprefent the ray after reflexion, for the fame reafons as above, as the angles O P T, F P E are equal, by Cor. 1. to Prop. XV. Book II. Hence it is evident, that if any number of rays proceed from one focus of an elliptic concave fpeculum, they will be reflected into the other. As rays of heat are fubject to the fame laws with rays of light, as to incidence and reflexion, if a fire or any heated body be placed at O, within the concave elliptic fpeculum, the whole of the heat after reflexion will meet at F. Perhaps attention

to

to this property might be of confiderable ufe in fitting up fire-places, reverberating furnaces, &c. BOOK III.

3. Let K P be a ray of light parallel to A B the axis of the parabola, by whofe revolution the parabolic fpeculum is generated, and let P F be drawn. Then will P F reprefent the ray after reflexion. For, K P being produced to D, the angle F P E will be equal to the angle D P E, by the Propofition preceding this Scholium, and therefore (15. i.) the angle K P T will be equal to the angle F P E. Confequently, as above, P F is the ray after reflexion. Hence if any number of rays, parallel to the axis, fall upon a concave parabolic fpeculum they will all after reflexion meet in the focus. Fig. 89.

The very confiderable magnifying powers which reflecting telefcopes are capable of, with parabolic fpecula, are to be attributed to this property. For a celeftial body being at an immenfe diftance, the rays which iffue from it upon the parabolic fpeculum are, as to fenfe, parallel to the axis; and, being all reflected to the focus, a diftinct and vivid image of the body is produced, provided the compofition of the metal be good and the parabolic figure juft.

PROP. X.

A ftraight line drawn from the focus of a parabola, perpendicular to a tangent, is a mean proportional between the ftraight line drawn from the point of contact to the focus, and the fegment of the axis between the focus and the vertex of the axis.

From F the focus of the parabola P B R let the ftraight line F G be drawn perpendicular to the ftraight line P E, touching the parabola in P, and draw P F; F G is a mean proportional between P F and F B, the fegment Fig. 89.

 of

BOOK III. of the axis A B, between F and B the vertex of the axis.

For let the tangent P E meet the axis in E, and draw B G, and let P A be an ordinate to the axis. Then, by Cor. Prop. IX. P F, F E are equal, and therefore (5. i.) the angles F P E, F E P are equal. Confequently in the triangles F G P, F G E, as the angles at G are right angles, P G is (26. i.) equal to G E, and the angles P F G, E F G are equal. Confequently, as, by Prop. V. A B is equal to B E, P G : G E : : A B : B E, and (2. vi.) G B is parallel to the ordinate P A, and therefore G B F is a right angle. The triangles P F G, G F B are therefore equiangular, and (4. vi.) P F : F G : : F G : F B.

The above Prop. is Lemma XIV. Lib. I. of the Principia.

Cor. 1. Hence (Cor. 2. 20. vi.) $PF^2 : FG^2$: : P F : F B.

Cor. 2. The concourfe of any tangent P E with a ftraight line F G, drawn from the focus of the parabola perpendicular to the tangent, is in the ftraight line B G, which touches the parabola in the vertex of the axis. For, by the above, G B is parallel to the ordinate P A, and therefore the Cor. is evident by Prop. II.

Cor. 3. If a ftraight line touch a parabola, and cut a ftraight line drawn from the focus to the directrix at right angles, it will bifect it. For, the reft remaining as above, let F G produced meet the directrix in D. Then as G B, D X are perpendicular to the axis, they are parallel, and (2. vi.) F B : B X : : F G : G D, and as, by Cor. 2. Def. X. F B is equal to B X, F G is equal to G D.

SCHOLIUM.

If a ftraight line pafs through a point moving in the curve of a conic fection, and always touch the fection, and

and if a ſtraight line revolve about a focus of the ſection as a center, and be always perpendicular to the moving tangent, the magnitude of the perpendicular will be leſs varied in the hyperbola than in the parabola, but it will be more varied in the ellipſe than in the parabola.

For let the point P be ſuppoſed to move in the curve B P of the conic ſection P B R, and let the ſtraight line P E accompany it in its motion, and always touch the ſection. Let the ſtraight line F G revolve about F, a focus of the ſection, and let it be always perpendicular to the tangent P E. Fig. 89. 90. 91.

In the ellipſe and hyperbola let C be the center, C D the ſemiconjugate axis, and C H the ſemidiameter parallel to the tangent P E. Then, by Prop. XIX. Book II. $FG : FP :: CD : CH$, and therefore, by Lemma V. $FG^2 : FP^2 :: CD^2 : CH^2$. But O being the other focus, and P O being drawn, by Prop. XVIII. Book II. $CH^2 = FP \times PO$, and therefore $FG^2 : FP^2 :: CD^2 : FP \times PO$, and $FG^2 = \frac{FP^2 \times CD^2}{FP \times PO} = \frac{FP \times CD^2}{PO}$. For the ſame reaſons if p denote another poſition of the moving point, Fg the perpendicular at that poſition, and Fp, pO ſtraight lines drawn from p to the foci, then $Fg^2 = \frac{Fp \times CD^2}{pO}$. Conſequently $FG^2 : Fg^2 :: \frac{FP \times CD^2}{PO} : \frac{Fp \times CD^2}{pO} :: \frac{FP}{PO} : \frac{Fp}{pO}$. Fig. 90. 91.

If in the parabola p denote another poſition of the moving point, Fg the perpendicular at that poſition, and pF a ſtraight line drawn to the focus, then, by the Propoſition preceding this Scholium, (and 17. vi.) $Fg^2 = Fp \times FB$, and $FG^2 : Fg^2 :: FP \times FB : Fp \times FB :: FP : Fp$; or $FG^2 : Fg^2 :: \frac{FP}{FB} : \frac{Fp}{FB}$. Fig. 89.

 In

BOOK III.

In the hyperbola and ellipſe, therefore, the ſquare of the perpendicular F G varies as the value of the fraction $\frac{\text{F P}}{\text{P O}}$ varies, and in the parabola it varies as $\frac{\text{F P}}{\text{F B}}$ varies.

But if the numerator and denominator of a fraction be each variable, then if they always increaſe or decreaſe in the ſame proportion, the value of the fraction will be always the ſame. Thus if the fraction be $\frac{Q}{R}$, and if while Q varies and becomes q, R varies and becomes r, and if it be $Q : R :: q : r$, then $\frac{Q}{R} = \frac{q}{r}$, by converting the proportion into an equation. From hence it is alſo evident, that the more nearly the numerator and denominator increaſe or decreaſe in the ſame proportion, the leſs will the value of the fraction be varied; but, on the contrary, the more they differ from a proportional increaſe or decreaſe, the more will the value of the fraction be varied. Now in the hyperbola the difference between F P, P O is, in every ſituation of P, equal to A B the tranſverſe axis, and therefore, in every inſtant they vary, they receive an equal increaſe or diminution; but however, in the parabola, F P may increaſe or diminiſh, F B remains conſtant. In the hyperbola therefore the value of the fraction $\frac{\text{F P}}{\text{P O}}$ varies leſs than the value of the fraction $\frac{\text{F P}}{\text{F B}}$ in the parabola. Again, in the ellipſe the ſum of P F, P O is equal to A B the tranſverſe axis, and therefore if either of them increaſe, the other will diminiſh; and conſequently in varying they will differ more from a proportional increaſe at the ſame time, or decreaſe at the ſame time, than F P, F B in the parabola. In the ellipſe

lipfe therefore the value of the fraction $\frac{FP}{PO}$ varies more than the value of the fraction $\frac{FP}{FB}$ in the parabola.

In the hyperbola therefore the fquare of F G varies lefs, but in the ellipfe it varies more, than it varies in the parabola; and confequently F G varies lefs in the hyperbola, but more in the ellipfe, than it varies in the parabola. See the Principia, Cor. 6. Prop. XVI. Lib. I.

PROP. XI.

If from a point in which a ftraight line touches a parabola two ftraight lines be drawn to the axis, one of them an ordinate to it, and the other at right angles to the tangent, the fegment of the axis intercepted between them will be equal to half the parameter of the axis: and if a ftraight line touching a parabola meet the axis, and a ftraight line, at right angles to the tangent, be drawn from the point of contact to the axis, the fegment of the axis intercepted between them will be equal to half the parameter of the diameter paffing through the point of contact.

Part I. From the point P, in which the ftraight line P E touches the parabola P B R, let the two ftraight lines P A, P H be drawn to A B the axis, one of them P A an ordinate to it, and the other P H at right angles to the tangent P E; the fegment H A of the axis, intercepted between them, is equal to half the parameter of the axis. Fig. 89.

For let the tangent P E meet the axis in E, and then as P A is an ordinate to the axis, it is at right angles to H E; and (Cor. 8. vi.) the fquare of P A is equal to the rectangle under E A, A H. But, by Prop. V. E B, B A are equal, and, by Prop. III. the fquare of P A is equal

 to

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to the rectangle under B A and the parameter of the axis. Consequently the rectangle under E A, A H is equal to the rectangle under B A and the parameter of the axis, and therefore (16. vi.) E A is to B A as the parameter of the axis to A H; and as B A is half of E A, A H must be half of the parameter of the axis.

Part II. Let the straight line P E touching the parabola P B R in P meet the axis A B in E, and let the straight line P H, at right angles to P E, meet the axis A B in H; the segment E H, intercepted between P E, P H, is equal to half the parameter of the diameter P K, passing through the point of contact.

For, the rest remaining as in the preceding part, the square of P E (Cor. 8. vi.) is equal to the rectangle under H E, E A; and, by Cor. 1. Prop. III. the square of P E is equal to the rectangle under E B and the parameter of P K. Consequently the rectangle under H E, E A is equal to the rectangle under E B and the parameter of P K, and therefore (16. vi.) E A is to E B as the parameter of P K to H E; and as E B is half of E A, H E must be equal to the half of the parameter of P K.

Cor. 1. The parameter of the axis is less than the parameter of any other diameter.

Cor. 2. A straight line drawn from any point in the curve of a parabola to the focus is equal to a fourth part of the parameter of the diameter passing through the same point. If the point be the vertex of the axis, this is evident from the seventh Definition; but for any other point P let every thing remain as in this Prop. and draw P F to the focus F. Then, by Cor. Prop. IX. F P, F E are equal; and as E P H is a right angle, if with F as a center, and F P as a distance, a circle be described, it will pass (31. iii.) through E and H. Consequently E F, F P, F H are equal, and therefore, by Part II. of this Prop. each of them is equal to a fourth part

part of the parameter of P K. This is Lemma XIII. Lib. I. of the Principia. BOOK III.

Cor. 3. The diftance of the vertex of any diameter of a parabola from the directrix is equal to a fourth part of the parameter of the diameter. This is evident from the preceding Cor. and Cor. 3. Prop. VIII.

PROP. XII.

Any ftraight line drawn through the focus of a parabola, and terminated both ways by the curve, is equal to the parameter of the diameter to which it is a double ordinate.

Let the ftraight line A H, paffing through F the focus of the parabola A B C, meet the curve in A and H, and be a double ordinate to the diameter D E; A H is equal to the parameter of D E. Fig. 87.

If D E be the axis, the Prop. is the fame as the Cor. to the feventh Definition. Let D E therefore not be the axis, and let it meet A H in E. Let D G touch the parabola in D the vertex of D E, and meet B F the axis in G, and draw D F. Then, by Cor. 1. to the firft Definition, and Prop. II. G E is a parallelogram, and therefore (34. i.) D E, G F are equal. But, by Cor. Prop. IX. G F, D F are equal, and, by Cor. 2. Prop. XI. D F is equal to a fourth part of the parameter of D E. Confequently D E is equal to a fourth part of the parameter of D E, and therefore, by Prop. III. AE^2 is equal to $DE \times 4DE$, and $4AE^2$ is equal to $4DE \times 4DE$. But (4. ii.) $4AE^2$ is equal to AH^2, and therefore A H is equal to 4 D E, and confequently equal to the parameter of D E.

Cor. If a ftraight line in a parabola pafs through the focus, and cut the diameter to which it is an ordinate, the abfcifs of the diameter will be equal to the diftance of

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of its vertex from the focus, and alſo from the directrix. For, as above, D E is equal to D F, and therefore, by Cor. 3. Prop. VIII. the remaining part of this is evident.

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Fig. 88. The ſuperficies A C L B I A, bounded by the ſtraight line A C and the curve A I B L C of a parabola, is called an *interior parabolic ſegment*; and if the ſtraight lines A E, C E, touching the parabola in A, C, meet one another in E, the ſuperficies bounded by A E, C E, and the curve A I B L C, is called the *exterior parabolic ſegment*, correſponding to the interior ſegment firſt mentioned.

XII.

If the ſtraight line R S, parallel to A C, touch the parabola in B, and meet in the points R, S the diameters A R, C S, the parallelogram A R S C is ſaid to be circumſcribed about the interior parabolic ſegment A I B L C A.

PROP. XIII.

In an interior parabolic ſegment a rectilineal figure may be inſcribed, and a correſponding rectilineal figure may be inſcribed in the exterior ſegment, ſo that the rectilineal figure in the interior ſegment ſhall be double of that inſcribed in the exterior; and theſe rectilineal inſcribed figures may be ſuch, that each ſhall be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than any given ſuperficies.

Fig. 88. Let A C L B I A be an interior, and E A I B L C E the correſponding exterior parabolic ſegment, as in the eleventh Definition; a rectilineal figure may be inſcribed in the ſegment A C L B I A, and a correſponding recti-

rectilineal figure may be inscribed in the segment E A I B L C E, so that the rectilineal figure in the interior segment shall be double of that in the exterior; and these rectilineal figures may be such, that each shall be less than the segment in which it is inscribed by a superficies less than a given superficies O.

For bisect A C in D, and draw E D, and let it cut the curve in B. Let the straight line F G touch the parabola in B, and meet the tangent E A in F, and the tangent E C in G, and draw A B, C B. Then, by Cor. 1. Prop. VI. E D is a diameter of the parabola, and, by Prop. II. F G, A C are parallel; and, by Prop. V. D B, B E are equal. Consequently (2. vi.) A F, F E are equal, and (1. vi.) the triangle A F B is equal to the triangle E F B; and therefore, as (1. vi.) the triangle A B D is equal to the triangle A E B, the triangle A B D is double of the triangle E F B. For the same reasons the triangle C B D is double of the triangle E G B; and therefore the triangle A B C, inscribed in the interior parabolic segment, is double of the triangle E F G, inscribed in the exterior parabolic segment. Again, bisect A B in H, and draw F H, and let it meet the curve in I. Let the straight line M N touch the parabola in I, and meet the tangent A F in M, and the tangent F B in N, and draw A I, I B. Then, as above, it may be proved, that the triangle A I B, inscribed in the interior parabolic segment A I B A, is double of the triangle F M N, inscribed in the exterior parabolic segment F A I B F. Also bisect B C in K, and draw G K, and let it meet the curve in L. Let the straight line P Q touch the parabola in L, and meet the tangent C G in Q, and the tangent B G in P, and draw B L, L C. Then it may be also proved, as above, that the triangle B L C, inscribed in the interior parabolic segment B L C B, is double of the

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triangle G P Q, inſcribed in the exterior parabolic ſegment G B L C G. Conſequently the rectilineal figure A I B L C, inſcribed in the interior parabolic ſegment firſt mentioned, is double of the rectilineal figure E M N P Q, inſcribed in the correſponding exterior parabolic ſegment firſt mentioned. In the ſame manner the inſcription in each ſegment may be continued, ſo that the whole figure inſcribed in the interior ſegment ſhall be double the whole figure inſcribed in the exterior ſegment.

The inſcription may alſo be continued till the rectilineal figure ſhall be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than the ſuperficies O. For in the interior ſegment the triangle A B C, being equal to half the triangle E A C, is greater than half the ſegment A I B L C A; and for the ſame reaſons the triangle A I B is greater than half the ſegment A I B A, and ſo of other remaining ſegments. Alſo in the exterior ſegment the triangle E F G, being half of the rectilineal figure E A B C E, is greater than half the ſegment E A I B L C E; and for the ſame reaſons the triangle F M N is greater than half the ſegment F A I B F, and ſo of other remaining ſegments. Conſequently (1. x.) the inſcription may be continued till each of the rectilineal figures ſhall be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than the given ſuperficies O.

Cor. The tangent P G being produced till it meet the diameters A R, C S in R and S, the circumſcribed parallelogram R A C S is equal to the triangle E A C, and therefore equal to the interior and exterior parabolic ſegments taken together. For, by the above, E F, F A are equal, and as R A, E B are parallel, the triangles E F B, A F R are equiangular, and (26. and 4. i.)

4. i.) therefore equal. For the ſame reaſons the triangles E G B, C G S are equal; and conſequently the Cor. is evident. BOOK III.

PROP. XIV.

An interior parabolic ſegment is double of the correſponding exterior parabolic ſegment; and the interior parabolic ſegment is to the circumſcribed parallelogram as two to three.

Every thing remaining as in the preceding Prop. and its Cor. the interior parabolic ſegment A C L B I A is double the correſponding exterior parabolic ſegment E A I B L C E, and the interior ſegment is to the circumſcribed parallelogram R A C S as two to three. Fig. 88.

Part I. If the interior ſegment be not double of the exterior ſegment, it muſt either be greater or leſs than its double. Firſt, let it be greater than the double of the exterior ſegment, and let the ſuperficies O be equal to the exceſs of the interior ſegment above the double of the exterior. Let correſponding rectilineal figures be inſcribed in the interior and exterior ſegments, as in the preceding Prop. and let them be ſuch that the rectilineal figure A I B L C, inſcribed in the interior ſegment, may be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than O, and let E M N P Q E be the correſponding rectilineal figure inſcribed in the exterior ſegment. Then the rectilineal figure A I B L C is greater than the double of the exterior ſegment, and therefore its half is greater than the exterior ſegment. But the rectilineal figure E M N P Q E is half of A I B L C, by Prop. XIII. and conſequently the rectilineal figure E M N P Q E is greater than the parabolic ſegment in which it is inſcribed: which is abſurd. The interior ſegment therefore cannot be greater than the double of the

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the exterior ſegment. Secondly, let the interior ſegment be leſs than the double of the exterior ſegment, and conſequently the exterior ſegment greater than half the interior ſegment. Let the exceſs of the exterior ſegment above half the interior be equal to the ſuperficies O; and let correſponding rectilineal figures be inſcribed in each ſegment, as in the preceding Prop. ſo that the rectilineal figure E M N P Q E inſcribed in the exterior ſegment may be leſs than the ſegment in which it is inſcribed by a ſuperficies leſs than O. Then the rectilineal figure E M N P Q E is greater than half the interior ſegment, and therefore its double is greater than the interior ſegment. Let A I B L C A, E M N P Q E be the correſponding rectilineal figures inſcribed in the two ſegments; and then, by Prop. XIII. as A I B L C A is double of E M N P Q E, it is greater than the interior ſegment in which it is inſcribed: which is abſurd. The interior ſegment is therefore not leſs than the double of the exterior ſegment. Conſequently as the interior ſegment is neither greater nor leſs than the double of the exterior, it is equal to the double of the exterior ſegment.

Part II. As, by Part I. the interior ſegment is double of the exterior, the interior ſegment is to the exterior as two to one, and therefore (18. V.) the interior ſegment is to the triangle E A C as two to three. Conſequently, by Cor. Prop. XIII. the interior ſegment is to the circumſcribed parallelogram R A C S as two to three *.

* Archimedes was the firſt who proved that a parabolic ſegment is equal to two thirds of the circumſcribed parallelogram. Of this truth he gave two demonſtrations. The firſt may be conſidered as mechanical, as it depends upon the primary properties of the lever. The ſecond is ſtrictly geometrical, and may be eaſily underſtood from the above; for he proves, that the triangle A B D is equal to four times the

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Fig. 92.

If A M N be the vertical plane to the oppofite hyperbolas E V F, G L H, as in the eighteenth Definition of the firft Book, and cut the cone in the fides A M, A N, and if planes A K, A I, touching the cone in the fides A N, A M, cut the plane of the hyperbolas in the ftraight lines K Q, I R, the ftraight lines K Q, I R are called the *Afymptotes* of either of the hyperbolas, or of the oppofite hyperbolas.

Cor. 1. As the vertical plane A M N is parallel to the plane of the hyperbolas E V F, G L H, the afymptotes K Q, I R (16. xi.) are parallel to A N, A M, fides of the cone.

Cor. 2. The afymptotes K Q, I R do not meet the curve of either of the oppofite hyperbolas. For the planes A K, A I touch the oppofite cones in the fides A N, A M, and as the afymptotes are parallel to thefe fides, they do not meet either of the oppofite conical fuperficies. The Cor. is therefore evident.

Cor. 3. Any two ftraight lines T S, P O drawn in the planes A K, A I from the afymptotes to the fides A N, A M, and parallel to the bafe of the cone, are equal, and touch the conical fuperficies. For let N M, I K, N K, M I be the lines of common fection of the bafe with the vertical plane, the plane of the hyperbolas,

the triangle A I B, each of thefe triangles being infcribed according to the conditions ftated above. The remainder of his demonftration is then equivalent to this, (fee Maclaurin's Algebra, §. 68.) that the fum of the infinite feries $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ &c. can neither be greater or lefs than $\frac{4}{3}$, the triangle A B C being analagous to 1.

The above Propofition and the foregoing remarks being clearly comprehended, the 5th Cor. to Lemma XI. of Sir Ifaac Newton's Doctrine of Prime and Ultimate Ratios will be eafily underftood.

BOOK III. and the planes A K, A I. Then, as Q K, S N are parallel, and as S T, being parallel to the baſe, is parallel to N K, the ſtraight line S T (34. i.) is equal to N K. For the ſame reaſons O P is equal to M I. Alſo N K, M I are equal. For if N K, M I be parallel, then M K is a parallelogram, and (34. i.) N K is equal to M I. But if N K be not parallel to M I, let them meet in W; and as they are in the tangent planes, they will touch the circle M D N, the baſe of the cone. The ſtraight lines W N, W M (36. iii.) are therefore equal, and (2. vi.) W N : N K : : W M : M I. Conſequently (14. v.) N K is equal to M I, and therefore S T, O P are equal, and as they are in the planes A K, A I, they touch the conical ſuperficies.

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The angle I C K, or Q C R, within which either of the oppoſite hyperbolas is ſituated, is called the *interior angle of the aſymptotes*; and the angle R C K, or Q C I, adjacent to it, is called the *exterior angle of the aſymptotes.*

PROP. XV.

The point in which the aſymptotes of an hyperbola cut one another is the center of the hyperbola; and any ſtraight line paſſing through the center and falling within the interior angle of the aſymptotes is a tranſverſe diameter; but any ſtraight line paſſing through the center and falling within the exterior angle of the aſymptotes is a ſecond diameter of the hyperbola.

Fig. 93. Part I. Let A B D, E F G be oppoſite hyperbolas, and let their aſymptotes I P, R N cut one another in C; C is the center of the hyperbola, or oppoſite hyperbolas.

In the curve of the hyperbola A B D take any two points A, D, and draw the ſtraight line A D. Then A D will

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Fig. 87.

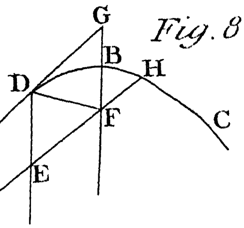


Fig. 82.

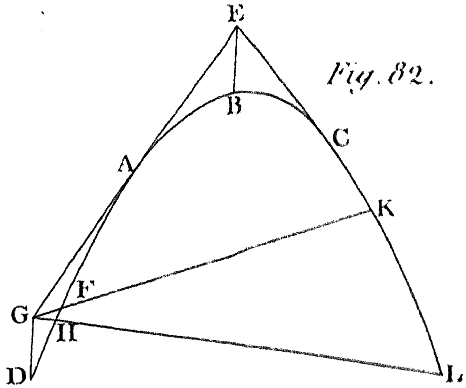
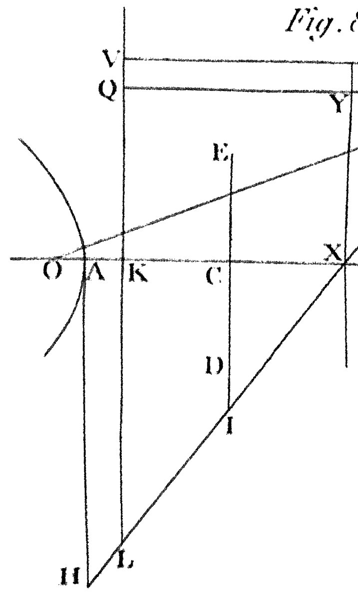


Fig. 8



.83.

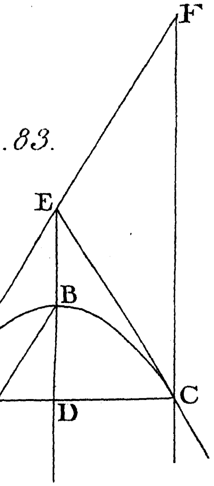
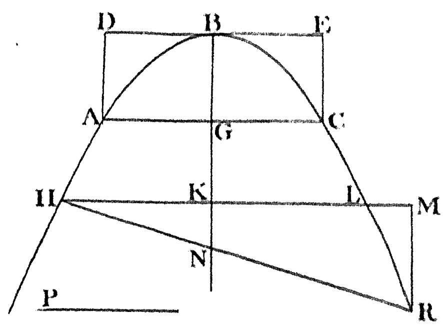


Fig. 81.



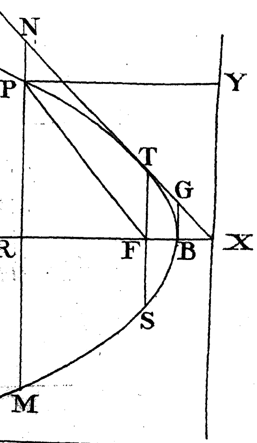


Fig. 85.

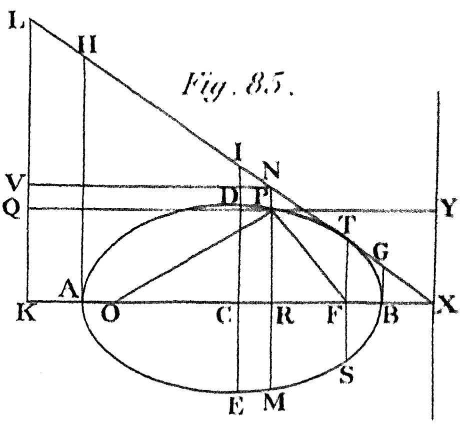
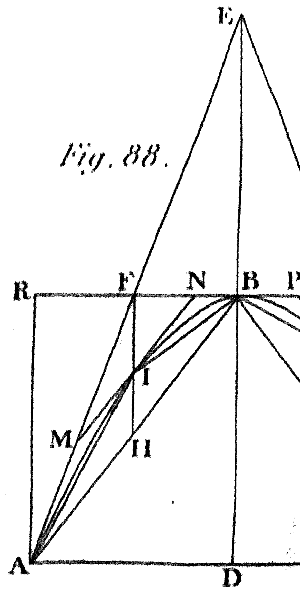


Fig. 88.



will meet both the asymptotes; for if it were parallel to either of the two, it would meet the curve in one point only, by Cor. 1. to the thirteenth Definition, and Prop. XIV. Book I. Let A D meet the asymptote R N in N, and the other in P; and let L M, parallel to A D, touch the hyperbola A B D in B, and meet R N in L, and I P in M. Let N V, L X, M Y, P W be straight lines parallel to the base of the cone in which the hyperbola was formed, and let them be supposed to have touched the conical superficies in the points V, X, Y, W; and then, by Cor. 3. to the thirteenth Definition, N V, L X, M Y, P W are equal. Also, by Cor. 1. Prop. XI. and Cor. 1. Prop. X. Book I. $BM^2 : MY^2 :: BL^2 : LX^2$; and $AP \times PD : PW^2 :: DN \times NA : NV^2$. Consequently (14. v.) BM^2 is equal to BL^2, and $AP \times PD$ is equal to $DN \times NA$; and therefore B M is equal to B L, and, by Lemma VI. P D is equal to N A. Draw C B, and let it meet A D in O; and then (4. vi.) $CB : CO :: BL : ON :: BM : OP$, and therefore (14. v.) O N is equal to O P. Consequently C B bisects A D, and for the same reasons it will bisect any straight line parallel to A D in the hyperbola; and therefore, by Cor. 1. Prop. III. Book II. C B is a diameter. Next, let the point H be in the curve of one hyperbola, and F in the other. Draw F H, and let it meet R N in Q, and I P in M. Let M Y, Q Z be two straight lines parallel to the base of the cone in which the sections were formed, and let them be supposed to have touched the conical superficies in Y and Z. Then, by Cor. 1. Prop. X. Book I. $HM \times MF : MY^2 :: HQ \times QF : QZ^2$, and therefore, by Cor. 3. to the thirteenth Definition (and 14. v.) $HM \times MF$ is equal to $HQ \times QF$, and, by Lemma VI. M H is equal to Q F. Consequently if C S be drawn bisecting Q M in S, it will bisect F H; and in the same manner, as above, it may be proved,

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proved, that C S will biſect any other ſtraight line G K parallel to F H, and terminated by the oppoſite curves. By Cor. 1. Prop. III. Book II. C S is therefore a diameter, and conſequently C is the center of the hyperbola.

Fig. 92. Part II. Every thing remaining as in the thirteenth Definition, any ſtraight line L V paſſing through the center C, and falling within the interior angle I C K of the aſymptotes, is a tranſverſe diameter of the hyperbola.

For draw A C, and through A C, L C V let a plane paſs, and let its line of common ſection with the vertical plane be A Z. Then (16. xi.) A Z, C V are parallel, as are alſo A N, C K; and therefore (10. xi.) the angles Z A N, V C K are equal. In the ſame manner it may be demonſtrated that the angles M A Z, I C V are equal, and that the angle M A N is equal to the angle I C K. The ſtraight line A Z therefore, paſſing through A the vertex of the cone, falls within the oppoſite ſuperficies, and conſequently, as in Part II. Prop. IX. Book I. it may be proved, that L C V meets the oppoſite conical ſuperficies. The ſtraight line L C V therefore, paſſing through C the center, meets the curves of the oppoſite hyperbolas, and conſequently is a tranſverſe diameter.

Fig. 93. Part III. Any ſtraight line C S, paſſing through C and falling within R C P, the exterior angle of the aſymptotes, is a ſecond diameter of the hyperbola.

For if through O, any point within the hyperbola A B D, a ſtraight line as A D be drawn parallel to C S, it will meet the aſymptotes, by Lemma III. and therefore, as it muſt cut the curve in two points, by Cor. 1. Prop. II. Book II. it is a ſecond diameter.

Fig. 93. *Cor.* 1. If a ſtraight line as L M touch an hyperbola and meet the aſymptotes, its ſegments between the point of contact and the aſymptotes will be equal. And

And if a ſtraight line cutting an hyperbola, or oppoſite hyperbolas, meet the aſymptotes, its ſegments between the curve or curves and the aſymptotes will be equal. This is evident from the above; for it was proved that N A is equal to P D, and H M to F Q.

Cor. 2. If through any point, as P, in either aſymptote, a ſtraight line, as K G, be drawn, meeting the oppoſite hyperbolas in K and G; and a ſtraight line, as P A, be drawn, meeting the curve of the hyperbola A B D in D and A; the rectangle under K P, P G will be equal to the ſquare of the ſemidiameter parallel to K G, and the rectangle under A P, P D will be equal to the ſquare of the ſemidiameter parallel to A P. For let E B be the diameter parallel to K G, and, the reſt remaining as above, let the ſtraight line C U be parallel to the baſe of the cone in which the ſection was formed, and let it be ſuppoſed to have touched the conical ſuperficies in U. Then, by Cor. 3. to the thirteenth Definition, P W, C U are equal, and, by Prop. XII. Book I. $KP \times PG : PW^2 :: BC \times CE$ or $CB^2 : CU^2$; and therefore, as PW^2, CU^2 are equal, $KP \times PG$ is equal to CB^2. Again, by Prop. V. Book II. $KP \times PG$ is to $AP \times PD$ as CB^2 to the ſquare of the ſemidiameter parallel to A P; and therefore (14. v.) the Cor. is evident.

Cor. 3. A ſtraight line as L M, touching the hyperbola A B D in B, and meeting the aſymptotes in L, M, is parallel and equal to the ſecond diameter conjugate to the tranſverſe diameter E B, paſſing through the point of contact. For it may be proved, as in the laſt Cor. that the ſquare of B M is equal to the ſquare of the ſemidiameter parallel to it. Conſequently, by Cor. 1. preceding, and by Cor. 3. Prop. III. Book II. this Cor. is evident.

BOOK III.

PROP. XVI.

According as a transverse diameter of an hyperbola is greater, equal to, or less than its conjugate diameter, the interior angle of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter is greater, equal to, or less than its conjugate diameter.

Fig. 94. Let G B be an hyperbola, of which C is the center, and C K, C I the asymptotes, and let A B be any transverse diameter, and D E the diameter conjugate to it; according as A B is greater, equal to, or less than D E, the interior angle K C I of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter F G is greater, equal to, or less than H L the diameter conjugate to it.

For let N I, touching the hyperbola in B, the vertex of A B, meet the asymptotes in N, I; and let K M, touching the hyperbola in G, the vertex of F G, meet the asymptotes in K, M. Then, by Cor. 1. and Cor. 2. Prop. XV. N I is bisected in B, and K M in G; and N I is equal to D E, and K M equal to H L, and therefore, according as A B is greater, equal to, or less than D E, C B is greater, equal to, or less than B I. But if with B as a center and B I as a distance a circle be described, its circumference will pass through N, and according as C B is greater, equal to, or less than B I, it will pass between C and B, through C, or on the opposite side of C from B. Consequently (by 31. iii. and 21. i.) according as C B is greater, equal to, or less than B I, the angle K C I is an acute, a right, or an obtuse angle. Again, if with G as a center and G M as a distance a circle be described, its circumference will pass through K, and (by 31. iii. and 21. i.) according as the angle K C I is an acute, a right, or an obtuse angle, the circumference of the circle will pass between C and G, through

through C, or on the oppofite fide of C from G. Confequently, according as K C I is an acute, a right, or an obtufe angle, C G is greater, equal to, or lefs than G M. The Propofition therefore is evident.

If two conjugate diameters of an hyperbola be equal, or if the angle contained by the afymptotes be a right one, it is called an *Equilateral Hyperbola.*

PROP. XVII.

The rectangle under two ftraight lines drawn from a point in the curve of an hyperbola to the afymptotes is equal to the rectangle under two ftraight lines, parallel to them, drawn from any other point in the curve of the fame or oppofite hyperbola to the afymptotes.

The rectangle under the two ftraight lines A E, A D, drawn from the point A in the curve of the hyperbola A V to the afymptotes C H, C K, is equal to the rectangle under the ftraight lines B F, B G, parallel to A E, A D, drawn from the point B in the curve of the fame or oppofite hyperbola to the afymptotes. Fig. 95.

For draw A B, and let it meet the afymptotes in H and K. Then, as A E, B F are parallel, (4. vi.) A E : B F : : H A : H B; and as B G, A D are parallel, B G : A D : : K B : K A. But, by Cor. 1. Prop. XV. H A is equal to K B, and therefore H B is equal to K A. Confequently (11. v.) A E : B F : : B G : A D*, and (16. vi.) A E × A D is equal to B F × B G.

Cor. 1. If A E, B F be parallel to the afymptote C K, and A D, B G be parallel to the afymptote C H; the parallelograms

* This is the property referred to by writers on Natural Philofophy, when they prove, that the curve formed by the upper furface of a liquid raifed by the force of attraction between two plates, meeting at one end, and kept at a fmall diftance from one another at the other, is an hyperbola.

BOOK III.

rallelograms E D, F G (14. vi.) are equal. For the angle at C being common to the two parallelograms, they are equiangular, and, by the above, the sides round the equal angles are reciprocally proportional.

Cor. 2. If from any two points as A, B in the curve of an hyperbola A V B two straight lines A E, B F, parallel to one of the asymptotes as C K, be drawn to the other asymptote C H; then C F : C E : : E A : F B. And the semitransverse diameters C A, C B being drawn, the triangles C F B, C E A are equal.

Cor. 3. If C H, C K be the asymptotes of an hyperbola A V B, and if from any two points F, E in C H, straight lines F B, E A be drawn parallel to C K, and if F B be drawn to the curve and E A towards it, and if E C be to C F as F B to E A, the point A must also be in the curve.

DEFINITIONS.

XV.

Fig. 96.

If from C the center of the hyperbola M N Q any two semitransverse diameters C N, C Q be drawn to the curve, the figure C N Q bounded by the semidiameters and the curve N Q is called a *Hyperbolic Sector.*

XVI.

If C R, C G be the asymptotes of the hyperbola M N Q, and from any two points N, Q in the curve, straight lines N B, Q A parallel to the asymptote C R be drawn to the other asymptote C G, the figure A B N Q, bounded by the straight lines N B, B A, A Q and the curve N Q, is called a *Hyperbolic Trapezium.*

Cor. A hyperbolic sector C N Q and trapezium A B N Q upon the same curve are equal. For let C N cut A Q in P, and then, as by Cor. 2. Prop. XVII. the triangles C A Q, C B N are equal, the rectilineal trapezium A B N P is equal to the triangle C P Q. To these equals add the

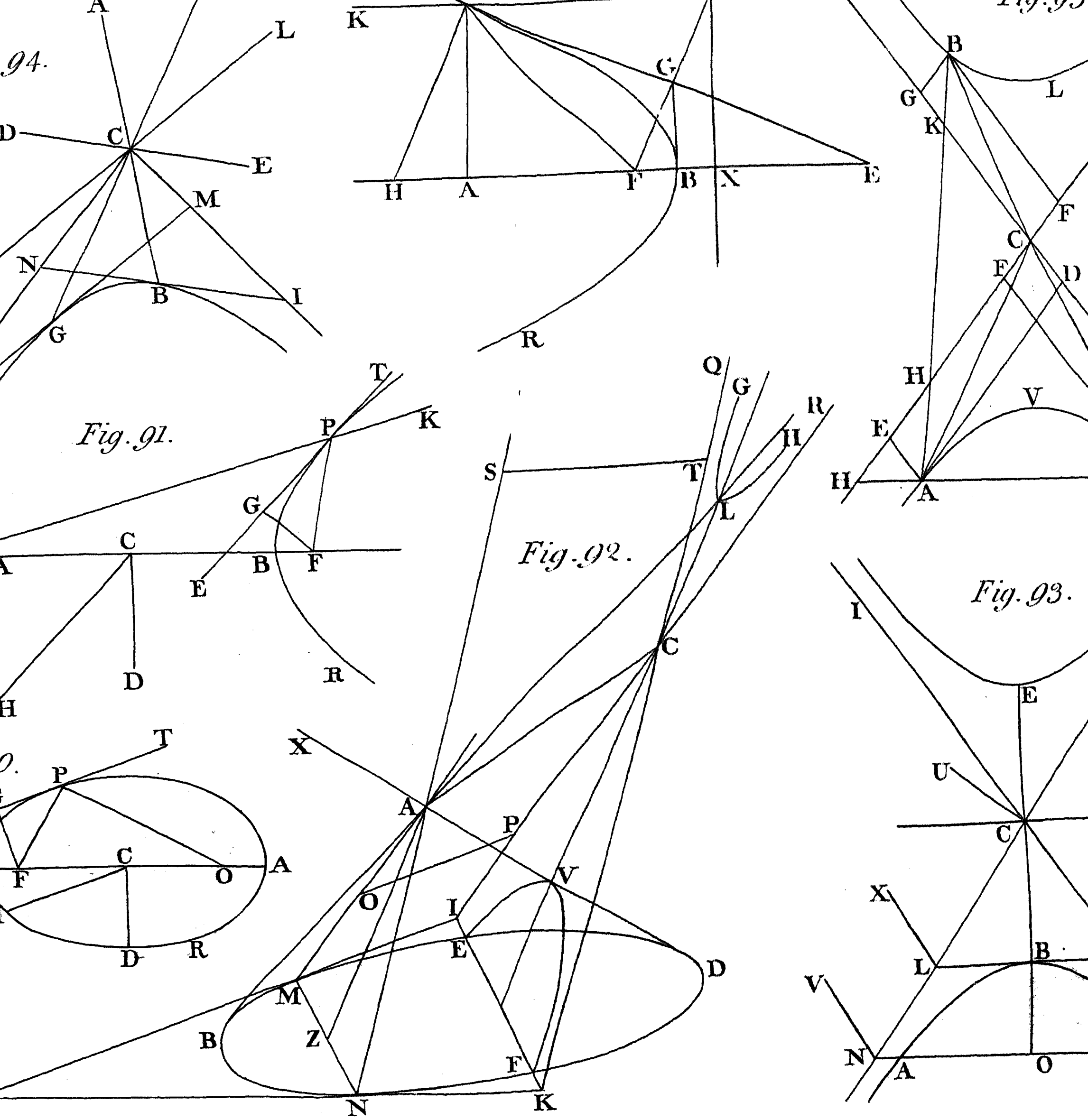


Fig. 94.

Fig. 95

Fig. 91.

Fig. 92.

Fig. 93.

0.

the figure P N Q, and then the hyperbolic ſector C N Q is equal to the hyperbolic trapezium A B N Q.

XVII.

Any ſegment as C A, intercepted between C the center and A a point in either aſymptote, is called an *Aſymptotic Segment*, and the point A is called its extremity.

XVIII.

Any ſtraight line in the plane of an hyperbola, or oppoſite hyperbolas, parallel to either of the aſymptotes is called an *Aſymptotic Secant*.

PROP. XVIII.

If from the points in which a ſtraight line cuts, and the point in which a ſtraight line parallel to it touches, an hyperbola, ſtraight lines parallel to one of the aſymptotes be drawn to the other, they will cut off from the center proportional aſymptotic ſegments: and, on the contrary, if from the extremities of three proportional aſymptotic ſegments ſtraight lines parallel to the other aſymptote be drawn to the curve of the hyperbola, the ſtraight line joining the extreme points in the curve will be parallel to the tangent paſſing through the middle point in the curve.

Let the ſtraight line M Q cut the hyperbola M N Q in the points M, Q, and let the ſtraight line G R, parallel to it, touch the hyperbola in N, and from Q, N, M let the ſtraight lines Q A, N B, M D, parallel to the aſymptote C R, be drawn to the aſymptote C G; the aſymptotic ſegments C A, C B, C D are proportionals. On the contrary, if C A, C B, C D be proportional aſymptotic ſegments, and ſtraight lines A Q, B N, D M, parallel to the aſymptote C R, be drawn to meet the curve of the hyperbola in Q, N, M, the ſtraight line Fig. 96.

BOOK III.

M Q, joining the extreme points, is parallel to G R touching the hyperbola in the middle point N.

Part I. Let the fecant M Q meet the afymptotes C G, C R in the points H, K, and let the tangent G R, parallel to the fecant, meet them in G, R. Then, as D M, A Q are parallel, (10. vi.) H M : H D : : K Q : C A, and therefore, by Cor. 1. Prop. XV. (and 14. v.) C A is equal to D H. Again, as by Cor. 1. Prop. XV. G N, N R are equal, and as B N, C R are parallel, (2. vi. and 14. v.) C B, B G are equal. Confequently C A : C B : : D H : B G; and (4. vi.) D H : B G : : D M : B N, and therefore (11. v.) C A : C B : : D M : B N. But, by Cor. 2. Prop. XVII. D M : B N : : C B : C D; and therefore (11. v.) C A : C B : : C B : C D.

Part II. Upon the fecond hypothefis let M Q meet the afymptotes in H, K, and let the tangent paffing through N meet them in G, R. Then, as above, it may be demonftrated that C A is equal to D H, and C B equal to B G; and therefore C A : C B : : D H : B G. But, by hypothefis, C A : C B : : C B : C D; and, by Cor. 2. Prop. XVII. C B : C D : : D M : B N. Confequently, (11. v.) D H : B G : : D M : B N, and by alteration D H : D M : : B G : B N; and therefore as D M, B N are parallel, the angles (6. vi.) D H M, B G N are equal, and (29. i.) H K, G R are parallel.

Cor. If two parallel ftraight lines M Q, L F cut an hyperbola L M Q F, and from the points F, Q, M, L ftraight lines F E, Q A, M D, L O parallel to the afymptote C T be drawn to the other afymptote C S, the afymptotic fegments C E, C A, C D, C O will be proportional. On the contrary, if the afymptotic fegments C E, C A, C D, C O be proportional, and E F, A Q, D M, L O, parallel to the afymptote C T, be drawn to the curve, the ftraight line L F joining the extreme points will be parallel to M Q joining the mean points. For, firft,

first, if G R, parallel to M Q or L F, touch the hyperbola in N and N B, parallel to the asymptote C T, be drawn to the other asymptote C S, then by the above (and 17. vi.) C A × C D is equal to CB^2, and also C E × C O is equal to CB^2. Consequently C E × C O is equal to C A × C D, and C E : C A :: C D : C O.

Upon the second hypothesis let C B be a mean proportional between C A, C D, and let B N, parallel to C T, be drawn to the curve, and let G R touch the hyperbola in N. Then, by the second Part of the above, G R, M Q are parallel. Again, as by hypothesis C E : C A :: C D : C O, C E × C O is equal to C A × C D. But C B being a mean proportional between C A, C D, CB^2 is equal to C A × C D, and therefore C E × C O is equal to CB^2, and C B is a mean proportional between C E, C O. Consequently as before L F is parallel to G R, and therefore (30. i.) M Q, L F are parallel.

PROP. XIX.

If from the extremities of four proportional asymptotic segments asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezium between the first and second secant will be equal to the hyperbolic trapezium between the third and fourth. And if from the extremities of a series of asymptotic segments, in geometrical progression, asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezia between the first and second secant, the first and third, the first and fourth, and so on, will be in arithmetical progression.

Part I. Let M N Q be an hyperbola, of which C is the center and C T, C S the asymptotes, and in C S let C E be to C A as C D to C O, and let E F, A Q, D M, O L be asymptotic secants drawn to the curve, the hyper- Fig. 96.

 bolic

BOOK III. bolic trapezium E A Q F is equal to the hyperbolic trapezium D O L M.

For M Q, L F being drawn, they will be parallel, by Cor. Prop. XVIII. Draw C L, C M, C Q, C F. Let C V be the diameter to which the parallels M Q, L F are double ordinates, and let it meet M Q in T, and L F in V. Then (38. i.) the triangle C L V is equal to the triangle C F V, and the triangle C M T is equal to the triangle C Q T; and as the diameter C V bisects all straight lines parallel to M Q and terminated by the curve, the space T M L V is equal to the space T Q F V. Consequently (axiom 3. i.) the hyperbolic sector C F Q is equal to the hyperbolic sector C M L; and therefore, by Cor. to the sixteenth Definition, the hyperbolic trapezia E A Q F, D O L M are equal.

Part II. The rest remaining as above, let C A, C B, C D, C X, &c. be a series of asymptotic segments in geometrical progression, and let the asymptotic secants A Q, B N, D M, X Y, &c. be drawn to the curve; the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progression.

For let G R touch the hyperbola in N, and then, by Prop. XVIII. it is parallel to M Q. Let the diameter C T pass through N, and then, by Prop. II. it bisects M Q in T; and (38. i.) the triangles C T Q, C T M are equal. And as C T bisects every straight line parallel to M Q, and terminated by the curve, the space N M T is equal to the space N Q T. Consequently (axiom 3. i.) the hyperbolic sectors C Q N, C N M are equal; and therefore, by Cor. to the sixteenth Definition, the hyperbolic trapezia A B N Q, B D M N are equal. As, by hypothesis, C B is to C D as C D to C X, it may be proved, in the same way, that the hyperbolic trapezia B D M N, D X Y M are equal; and the same mode of proof may be extended to any number of terms. Consequently

ſequently the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progreſſion.

SCHOLIUM.

As the hyperbola and its aſymptotes may be indefinitely extended, it is evident that a ſeries of aſymptotic ſegments in geometrical progreſſion, and a correſponding ſeries of hyperbolic trapezia in arithmetical progreſſion, may be continued to any aſſigned number of terms. From the nature of logarithms, therefore, the ſeries of aſymptotic ſegments C A, C B, C D, &c. as above, is analogous to a ſeries of natural numbers in geometrical progreſſion, and the ſeries of hyperbolic trapezia A B N Q, A D M Q, &c. as above, is analogous to the logarithms of theſe natural numbers. To enter into an explanation of theſe analogies would be incompatible with the deſign of this work. The reader may find full information on the ſubject of logarithms in the volumes entitled, "Scriptores Logarithmici," publiſhed by Francis Maſeres, Eſq. F. R. S. Curſitor Baron of the Court of Exchequer. To this Gentleman the mathematical world are highly indebted for his perſevering exertions and liberality in the cauſe of ſcience. He has employed his great abilities in endeavours to render ſome of the moſt important ſubjects perſpicuous, and he has expended large ſums in the publication of ſcarce mathematical tracts, and made preſents of many copies of them, with the highly laudable motive of promoting learning and diſſeminating knowledge.

Within theſe twenty years, laſt paſt, much has been done in this country to facilitate the application of logarithms, and to extend their utility. In 1785 Dr. Hutton of Woolwich publiſhed in 8vo. extenſive Tables of them, to which he prefixed "A large and original Hiſtory of the Diſcoveries and Writings relating to

BOOK III. to thofe fubjects." Thefe Tables are judicioufly arranged, and are very valuable for general ufe. The hiftory prefixed to them is a mafterly performance; it gave rife to the publication mentioned above, entitled, "Scriptores Logarithmici."

In the year 1792 a quarto volume was publifhed, under the patronage of the Board of Longitude, entitled, "Tables of Logarithms of all Numbers, from 1 to 101000; and of the Sines and Tangents to every fecond of the Quadrant. By Michael Taylor, Author of the Sexagefimal Table." As Mr. Taylor died before the Tables were entirely printed, the Rev. Dr. Mafkelyne, Aftronomer Royal, fuperintended their completion. He alfo wrote the Preface and Precepts for the ufe of the Tables; and thefe he executed with that care, learning, and ability, for which he is fo juftly celebrated in every part of the world where either the theory, or practical utility, of Aftronomy is underftood. As Taylor's Tables are accurate, and more extenfive than any other extant, the volume is an excellent refource to thofe who aim at a fuperior degree of precifion in their calculations.

In a fmall quarto volume of mathematical Effays, publifhed in 1788 by the Rev. John Hellins, (now Vicar of Potters' Pury, Northamptonfhire, and F.R.S.) there are two Effays on Logarithms, which difplay an intimate knowledge of the fubject. A fcientific reader will find much gratification in the perufal of thefe Effays.

DEFINITION.
XIX.

Fig. 97. If A B be a tranfverfe diameter of the oppofite hyperbolas A, B, and D E the fecond diameter conjugate to it, and if D E be a tranfverfe diameter of the oppofite

ſite hyperbolas D, E, and A B the ſecond diameter conjugate to it; the hyperbolas D, E are called the *Conjugate Hyperbolas* to one or both of the oppoſite hyperbolas A, B, and, on the contrary, the hyperbolas A, B are called the *Conjugate Hyperbolas* to one or both of the oppoſite hyperbolas D, E. When all the four hyperbolas A, D, B, E are mentioned together, they are called *Conjugate Hyperbolas.*

Cor. If the diameters A B, D E cut one another in C, it is evident that C is the common center of the conjugate hyperbolas.

PROP. XX.

One of the aſymptotes of an hyperbola is parallel to, and the other biſects, a ſtraight line joining the vertices of any two conjugate diameters: and the vertices of ſecond diameters of an hyperbola are in the curves of the hyperbolas conjugate to it.

Part I. Let A B, D E be any two conjugate diameters of the hyperbola A H, and let C F, C G be its aſymptotes, C being the center; one of the aſymptotes, as C F, is parallel to A E the ſtraight line joining the vertices A, E, and the other aſymptote C G biſects A E. Fig. 97.

For let F G touch the hyperbola H A in A and meet the aſymptotes in F and G. Then, by Cor. 3. and 1. Prop. XV. F G is equal and parallel to D E, and C E is equal to F A and alſo to A G. Conſequently (33. i.) the aſymptote C F is parallel to A E. Alſo the angle (29. i.) C E L is equal to the angle L A G, and the angle E C L to the angle A G L, and therefore (26. i.) A L is equal to E L, and A E is biſected by the aſymptote C G.

Part II. Let K H be any tranſverſe diameter of the oppoſite hyperbolas A H, B K, and let M N be the ſecond

BOOK III. cond diameter conjugate to it; the vertices M, N of this fecond diameter are in the curves of the hyperbolas conjugate to the hyperbolas A H, B K.

For let D, E be the hyperbolas conjugate to A H, B K, and let A B, D E be the conjugate diameters common to the conjugate hyperbolas, as in the nineteenth Definition, and C the center. Let F C, G C be the afymptotes of the oppofite hyperbolas A, B. Draw B E, K N, and let them meet the afymptote F C in P and Q; and then, by Part I. B E, K N are parallel to the afymptote G C, and they are bifected by the afymptote F C in P and Q. Confequently, by Cor. 2. Prop. XVII. C P : C Q : : K Q : B P; and therefore, on account of the equals, C P : C Q : : Q N : P E, and as E is in the curve of the hyperbola E, N muft be in the curve of the fame hyperbola E, by Cor. 3. Prop. XVII.

Cor. A ftraight line parallel to one of the afymptotes, and terminated by the curves of the conjugate hyperbolas, is bifected by the other afymptote.

PROP. XXI.

A quadrilateral figure, whofe fides pafs through the vertices of any two conjugate diameters of an ellipfe, or conjugate hyperbolas, and touch the ellipfe or hyperbolas, is a parallelogram, and is equal to the rectangle under the axes of the ellipfe or hyperbolas.

Fig. 98. 99. Let M S L R be a quadrilateral figure, whofe fides M S, S L, L R, R M pafs through F, G, H, K, the vertices of the conjugate diameters F H, G K of the ellipfe F G H K, or the conjugate hyperbolas F, G, H, K, and in thefe points touch the curve or curves; the figure M S L R is a parallelogram, and is equal to the rectangle under the axes A B, D E of the ellipfe or hyperbolas.

For,

For, by Cor. 2. Prop. IV. Book II. the tangents M S, R L are parallel to the diameter K G, and the tangents M R, L S are parallel to the diameter F H. The quadrilateral figure M S L R is therefore a parallelogram.

Again, let C be the center, and let C I be perpendicular to the tangent S L. Then as F H is bifected in C, the parallelogram G H is a fourth part of the parallelogram L M; and, by Cor. 1. Prop. XIX. Book II. C I : C B : : C D : C H, and C I × C H is equal to C B × C D. But (35. i.) C I × C H is equal to the parallelogram G H, and C B × C D is a fourth part of the rectangle under the axes A B, D E. Confequently the parallelogram L M is equal to the rectangle under the axes A B, D E.

Cor. All parallelograms contained under tangents, paffing through vertices of conjugate diameters of an ellipfe, or conjugate hyperbolas, are equal to one another, and each of them is equal to the rectangle under the axes.

The twelfth Lemma of Sir Ifaac Newton's Principia, Lib. I. and the tenth Propofition, depending upon it, are evident from the above.

PROP. XXII.

If through a point in the curve of an hyperbola two ftraight lines be drawn parallel to the afymptotes and meeting a diameter, the femidiameter will be a mean proportional between the fegments of the diameter between the center and the points of concourfe.

Through the point I in the curve of the hyperbola I B let the ftraight lines I T, I A be drawn parallel to the afymptotes C E, C G, and firft let them meet the tranfverfe diameter D B in the points T, A, and let C be the Fig. 100.

BOOK III. the center; the femidiameter C B is a mean proportional between C T, C A.

For let I A meet the afymptote C E in F, and let B H, T K be parallel to I F, or to the afymptote C G, and let them meet C E in H and K. Then (34. i.) K T, F I are equal; and (4. vi.) C K : C H : : K T or its equal F I : H B. But, by Cor. 2. Prop. XVII. F I : H B : : C H : C F, and therefore (11. v.) C K : C H : : C H : C F. Confequently, on account of the parallels K T, H B, F A, C T : C B : : C B : C A.

Secondly, the reft remaining as above, let the ftraight lines I T, I A meet the fecond diameter L M in the points N, O, and let I A meet the curve of the hyperbola P L, conjugate to B I, in P; and let P Q be parallel to C E or I N, and meet the diameter L M in Q. Then, by the above, C Q : C L : : C L : C O. But, by Cor. Prop. XX. I F, F P are equal, and therefore, as P Q, I N are parallel, C Q is equal to C N. Confequently C N : C L : : C L : C O.

Fig. 101. 102. *Cor.* 1. If two ftraight lines T N, T P, touching an hyperbola, or oppofite hyperbolas, in N and P, meet one another in T, and if a ftraight line T I parallel to one of the afymptotes meet the curve in I, a ftraight line I A parallel to the other afymptote and meeting N P, the ftraight line joining the points of contact, in A will bifect N P in A. This is evident from the above, and Prop. VII. Book II.

Cor. 2. The reft remaining as in the preceding Cor. if T I produced meet N P in E, T E will be bifected in I. For let T L parallel to I A, or an afymptote, meet the curve in L, and then, by Cor. 1. L A parallel to T I will meet N P in A, the point in which N P is bifected; and T I A L is a parallelogram. Draw I L, and let it meet the diameter C B A in G. Then (34. i.) I A, T L are equal, and (29. i.) the triangles A I G, T L G are equi-

SC.

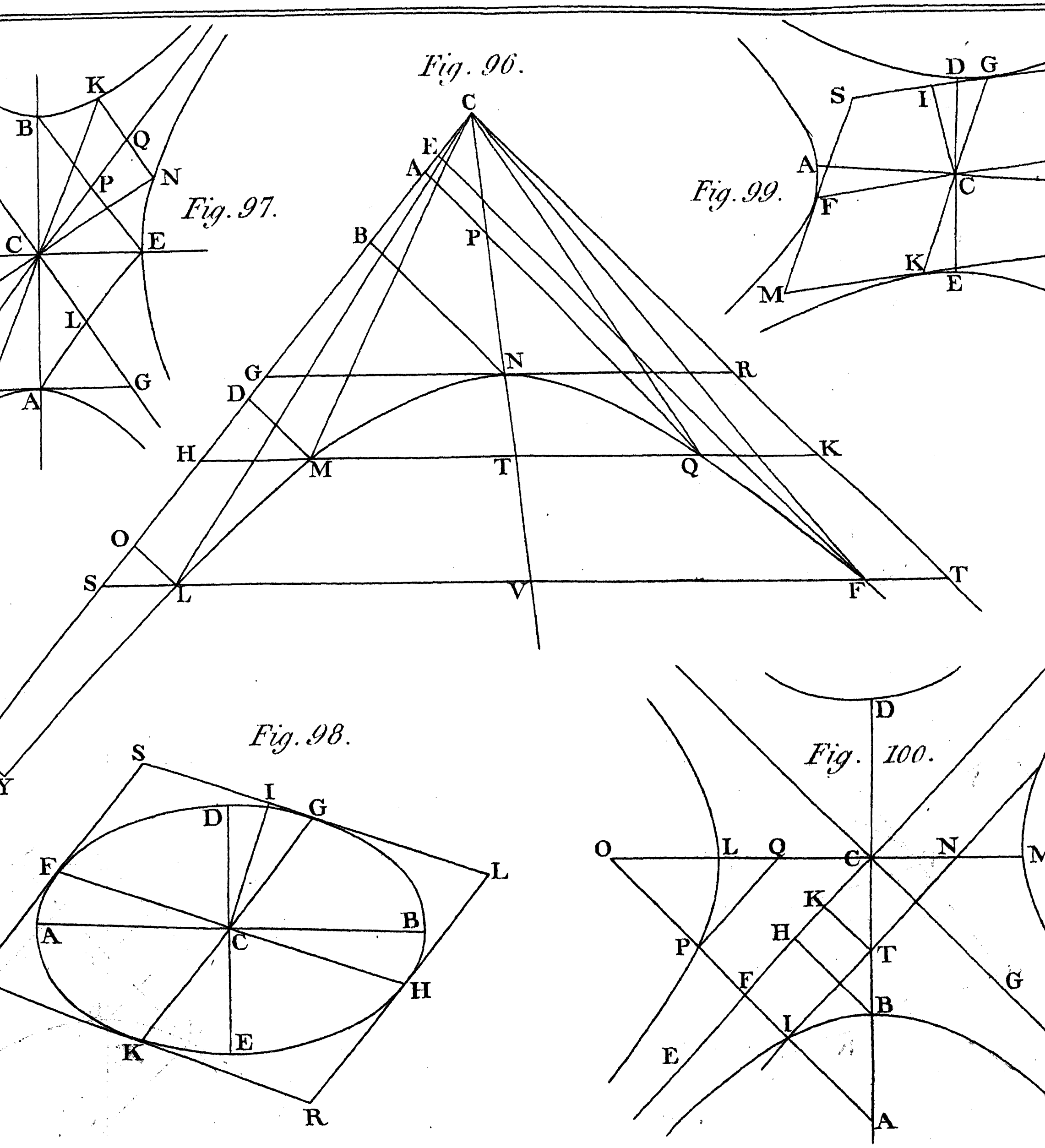


Fig. 96.

Fig. 97.

Fig. 98.

Fig. 99.

Fig. 100.

equiangular; and therefore (26. i.) I G, G L are equal, and T G is equal to G A. The ſtraight lines I L, N P are therefore ordinates to the diameter C B A, and conſequently, by Prop. II. they are parallel; and (2. vi.) T G : G A :: T I : I E. The ſtraight line T E is therefore biſected in I.

PROP. XXIII.

If a diameter of a parabola, or a ſtraight line parallel to an aſymptote of an hyperbola, meet two tangents and the ſtraight lines joining the points of contact, the ſquare of its ſegment between the curve and the ſtraight line joining the points of contact will be equal to the rectangle under the ſegments between the curve and tangents.

If the diameter of the parabola, or the ſtraight line parallel to an aſymptote of an hyperbola, paſs through the point in which the tangents meet one another, the Propoſition is evident from Prop. V. and Cor. 2. Prop. XXII. but if not, let the two ſtraight lines M N, M O touch the parabola, hyperbola, or oppoſite hyperbolas, in the points N, O, and let T X a diameter of the parabola, or a ſtraight line parallel to an aſymptote, meet the parabola or either hyperbola in E, the tangents in A, D, and the ſtraight line joining the points of contact in B; the ſquare of E B is equal to the rectangle under A E, E D. Fig. 103. 104. 105.

Caſe 1. If the ſtraight lines touching the parabola or hyperbola meet one another in M, from the point D in which T X meets one of the tangents draw D L parallel to the other tangent M O, and let it meet the curve in P, L, and the ſtraight line N O in K. Then, by Prop. XVII. Book I. the ſquare of D K is equal to the rectangle under L D, D P; and (4. and 22. vi.) Fig. 103. 104.

A B²

BOOK III. $AB^2 : DB^2 :: AO^2 : DK^2$ or $LD \times DP$. But as TX is parallel to a ſide of the cone in which the ſection was formed, by Prop. XVI. Book I. $AO^2 : LD \times DP ::$ $AE : DE$; and therefore (11. v.) $AE : DE :: AB^2 :$ DB^2. Conſequently, by Lemma VIII. $AE : BE ::$ $BE : DE$; and (17. vi.) $AE \times ED$ is equal to BE^2.

Fig. 105. Caſe 2. If the ſtraight lines touching the oppoſite hyperbolas meet one another in M, from A or D, ſuppoſe D, draw the ſtraight line DK parallel to the tangent AO, and let it meet NO in K. Through the points A, D draw the ſtraight lines GH, PL parallel to one another, and let them meet the oppoſite hyperbolas in the points G, H and P, L. Then, by Prop. V. Book II. $GA \times AH$ is to AO^2 as the ſquare of the ſemidiameter parallel to GH to the ſquare of the ſemidiameter parallel to AO. And, by Cor. 1. Prop. XVII. Book I. and Prop. V. Book II. $PD \times DL$ is to DK^2 as the ſquares of the ſame ſemidiameters. Conſequently (11. v. and alternation) $GA \times AH : PD \times DL :: AO^2 :$ DK^2. But (4. and 22. vi.) $AO^2 : DK^2 :: AB^2 : DB^2$; and, by Prop. XVI. Book I. $GA \times AH : PD \times DL ::$ $AE : DE$. Conſequently $AE : DE :: AB^2 : DB^2$, and therefore, by Lemma VIII. $AE : BE :: BE : DE$, and $AE \times DE$ is equal to BE^2.

Fig. 105. Caſe 3. If the ſtraight lines AO, DN touching the oppoſite hyperbolas be parallel, then the triangles ABO, DBN will be equiangular, and the proportion will be $AO^2 : DN^2 :: AB^2 : DB^2$; and in this caſe, by Prop. XVI. Book I. $AE : DE :: AO^2 : DN^2$. Conſequently $AE : DE :: AB^2 : DB^2$, and, by Lemma VIII. $AE :$ $BE :: BE : DE$, and $AE \times DE$ is equal to BE^2.

PROP. XXIV.

If from two given points in the curve of a parabola, or hyperbola, or the curves of oppoſite hyperbolas, two ſtraight

ſtraight lines be inflected to any third point in the curve of the parabola, or in the curve of either of the oppoſite hyperbolas, and if they meet a diameter of the parabola, or a ſtraight line parallel to an aſymptote of the hyperbola; the ſegments of this laſt mentioned line, between the inflected lines and the point in which it meets the curve of the ſection, will be to one another in the ſame ratio, wherever the point may be in the curve to which the lines are inflected.

Let N, M be two given points in the curve of the parabola, hyperbola, or oppoſite hyperbolas, and let the ſtraight lines N O, M O inflected from them to any point O in the curve of the parabola, or in the curve of either hyperbola, meet in B, C the ſtraight line T X, a diameter of the parabola, or parallel to an aſymptote of the hyperbola, and let T X meet the curve in E; the ſegments E B, E C are to one another in the ſame ratio, wherever the point O may be taken in the curve. Fig. 106. 107. 108.

For let tangents paſſing through M, N, O meet T X in F, D, A. Draw M N, and let it meet T X in G; and by Prop. XXIII. $ED : EG :: EG : EF$. Again, by Prop. XXIII. EB^2 is equal to $AE \times ED$, and EC^2 is equal to $AE \times EF$. Conſequently $EB^2 : EC^2 :: AE \times ED : AE \times EF ::$ (1. vi.) $ED : EF$. But, by the above, (and Cor. 2. 20. vi.) $ED : EF :: ED^2 : EG^2$; and therefore (11. v.) $EB^2 : EC^2 :: ED^2 : EG^2$, and (22. vi.) $EB : EC :: ED : EG$. But as the points M, N are given, or fixed, and as the ſtraight line T X is given in poſition, the ſegments E D, E G remain the ſame. Conſequently, if the point O be moved round the curve of the parabola, hyperbola, or oppoſite hyperbola, the ſegments E B, E C of the ſtraight line T X, between the inflected lines and the curve, will be to one another in the ſame ratio, wherever the point O may be in the curve.

BOOK III.

SCHOLIUM.

It may be proper in this place to direct the attention of the reader to methods of ascertaining certain particulars in a conic section, supposing the curves of the sections to be given, or straight lines to be given, for the description of the curves. These methods, now to be described, might have been delivered as Corollaries to the Propositions on which they depend, or they might have been put into the form of Problems; but it appeared more adviseable to reserve them for a series of articles in a Scholium. A cautious desire against interrupting the reader in the acquisition of new truths suggested this delay. The ease with which they are deduced from the preceding Propositions, and the importance of the articles themselves, induced the author to think that the following was the most proper manner of delivering them, and the most proper place to insert them.

Fig. 109. 110. 1. Let the ellipse A D B E, or the opposite hyperbolas A, D B E, be given to find the center.

In the ellipse and in either hyperbola draw the two parallel straight lines D E, F G, and draw A B bisecting D E in H and F G in K. Let A B meet the curve of the ellipse, or the curves of the opposite hyperbolas, in A and B. Bisect A B in C, and C will be the center, by Cor. 1. Prop. III. Book II.

If only one hyperbola D B E be given, two other straight lines must be drawn parallel to one another, but not parallel to D E, F G, and a straight line being drawn bisecting them will be a diameter. Its concourse therefore with A B will determine the center.

2. The curve of a conic section and a point in it being given, let be required to draw a diameter through the given point.

If the section be an ellipse, or hyperbola, find the cen-

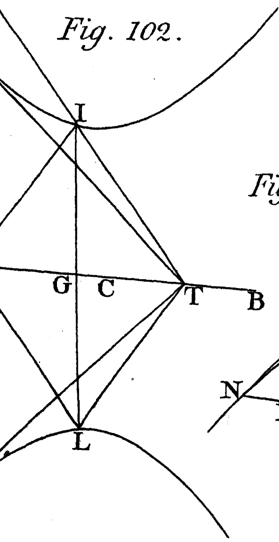


Fig. 102.

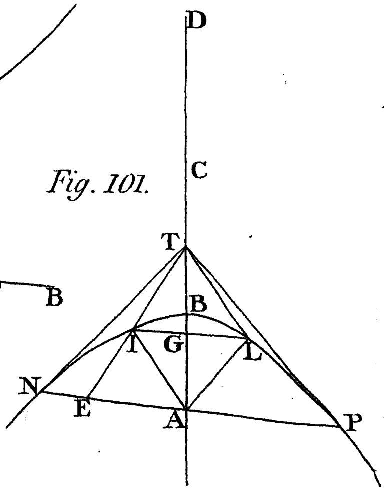


Fig. 101.

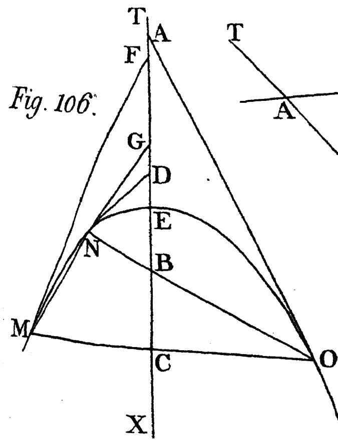


Fig. 106.

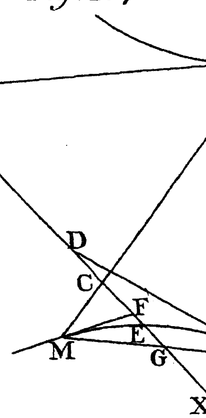


Fig. 107.

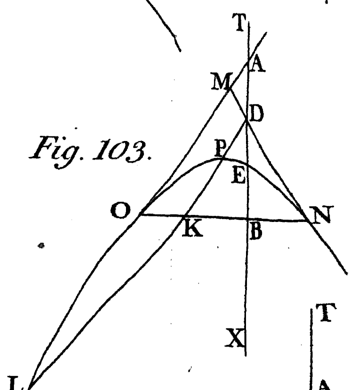


Fig. 103.

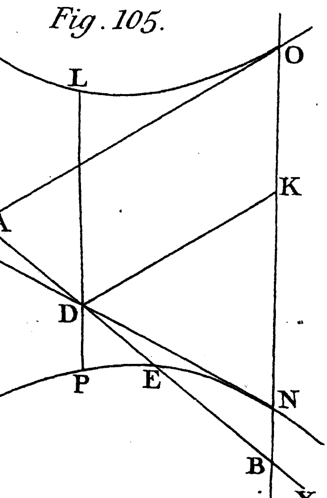


Fig. 105.

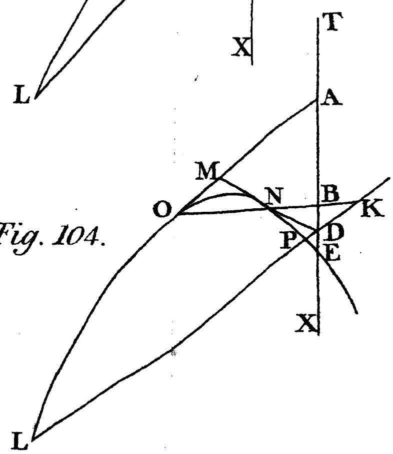


Fig. 104.

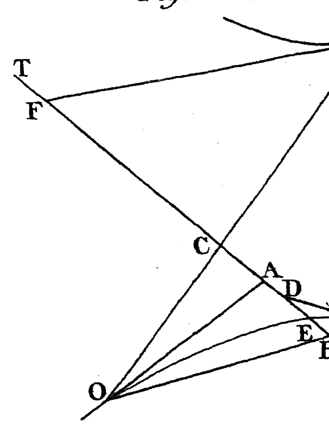


Fig. 108.

center by the preceding article, and through the center and the given point draw a diameter. If the section be a parabola, find a diameter, by Cor. 3. Prop. II. of this Book, and parallel to it draw a ſtraight line through the given point; and, by Cor. 1. to the firſt Definition, this will be the diameter required.

3. The curve and a diameter A B of a conic ſection being given, and any point G in the curve beſides a vertex of the given diameter, let it be required to draw a ſtraight line from G ordinately applied to the diameter.

Firſt, let the ſection A G B D be an ellipſe. Find the center C by the firſt article, and through it draw the diameter G L. Through the vertex L draw the ſtraight line L F parallel to A B. Then if L F touch the ellipſe, A B, L G will be conjugate diameters, by Cor. 2. Prop. IV. Book II. and L G will be ordinately applied to A B. But if L F do not touch the ellipſe, let it meet the curve again in F. Draw G F, and let it meet A B in K, and G F will be the ordinate required. For, as C K, L F are parallel, (2. vi.) G C : G K : : C L : K F, and therefore (14. v.) G F is biſected in K. Secondly, let the ſection F B G be an hyperbola, or parabola. In the diameter A B take any point M, and the ſtraight line G M being drawn, produce it to L, ſo that M L may be equal to G M. Then if the point L be in the curve, G L will be the ordinate required; but if L be not in the curve, draw the ſtraight line L F parallel to A B, and let it meet the curve in F, according to Prop. IX. Book I. or Cor. 1. Def. I. of this. Draw G F, and it will be the ordinate required. For, if it meet A B in K, it may be proved, as above, that G F is biſected in K. Fig. 109. Fig. 110.

4. The curve of a conic ſection F B G being given, and a point B being given in it, let it be required to draw a ſtraight line through B to touch the ſection. Fig. 110.

 Through

BOOK III.

Through B draw A B, a diameter of the fection, and, by the laft article, draw G F an ordinate to it. Through the vertex B draw the ftraight line T B parallel to G F, and, by Prop. II. T B will be the tangent required.

Fig. 111. 5. Two unequal ftraight lines A B, D E being given, bifecting one another in C at right angles, let it be required to defcribe the curve of an ellipfe, of which A B, D E fhall be the axes, and C the center.

Let A B be greater than D E, and confequently the tranfverfe axis. Find the foci F, O in A B, by Cor. 2. Def. XI. Book II. Let the ends of a thread or ftring, equal in length to A B, be fixed to the points F, O. By means of the pin P let the thread or ftring be ftretched; and, while it continues uniformly tenfe, let the end or point of the pin P move round in the plane, in which A B, D E are fituated, till it return to the fame place from which it began to move. The line traced by the end or point of the pin P is the curve of an ellipfe, as is evident from Prop. XIII. Book II. and A B, D E are the axes.

Fig. 112. 6. Two ftraight lines A B, D E being given, bifecting one another in C at right angles, let it be required to defcribe the curve of an hyperbola, of which A B fhall be the tranfverfe and D E the conjugate axes.

In A B produced both ways let the foci F, O be found, by Cor. 3. Def. XI. Book II. Let one end of a thread or ftring F P R be fixed to the point F, and let the other end be fixed to the extremity R of the ruler O R, and let the length of the ruler exceed the length of the thread or ftring by the ftraight line A B. Let O, the other extremity of the ruler, be fixed to the point O, and let the ruler revolve about O as a center. By means of the pin P let the thread or ftring be ftretched, and let the part between P and R be kept clofe to the edge of the ruler; and while the ruler revolves,

in the plane in which A B, D E are ſituated. The line G B H will be the curve of an hyperbola, of which A B is the tranſverſe and D E is the conjugate axes, and C is the center, as is evident from Prop. XIII. Book II.

7. The ſtraight line D X, of indefinite length, being Fig. 113.
given, and F being a point given without it, let it be required to deſcribe the curve of a parabola, of which D X ſhall be the directrix and F the focus.

Place the edge of a ruler R D X L along the line D X, and keep it fixed in that poſition. Let G Y E be a ruler of ſuch a form that the part G Y may ſlide along the edge D X of the fixed ruler R D X L, and the part Y E may be always perpendicular to D X. Let E P F be a thread or ſtring of the ſame length with the part Y E of the moving ruler, and let one end of it be fixed to the ruler at E, and let the other end be fixed to the point F. By means of the pin P let the thread or ſtring be ſtretched, and the part between P and E be kept cloſe to the edge of the ruler. While the ruler G Y E ſlides along the edge D X of the fixed ruler, and the thread or ſtring is kept uniformly tenſe, let the end or point of the pin P trace the line A P B C on the plane, in which the line D X and the point F are ſituated. The line A P B C will be the curve of a parabola, of which D X is the directrix and F the focus, as is evident from Cor. 3. Prop. VIII.

Several writers on Conic Sections have defined the ellipſe, hyperbola, and parabola by the deſcription in Article 5, 6, and 7 reſpectively; and from theſe deſcriptions, as founded on a primary, they have deduced other properties of the ſections.

8. Two ſtraight lines A B, D E being given, biſecting Fig. 114.
one another in C but not at right angles, let it be 115.

BOOK III.

required to deſcribe an ellipſe, or hyperbola, of which A B, D E ſhall be conjugate diameters, and C the center.

In C D, produced in the ellipſe but between C and D in the hyperbola, take the point N, ſo that the rectangle under C D, D N may be equal to the ſquare of C B. Through D draw the ſtraight line M Q parallel to A B, and biſect C N in I. Draw I P perpendicular to C N, and let it meet M Q in P; and then it is evident (4. i.) that ſtraight lines drawn from P to N and C will be equal. With P as a center therefore, and P N or P C as a diſtance, let the circle M C Q N be deſcribed, and let it meet the ſtraight line M Q in M and Q. Draw the ſtraight lines Q C, M C; and from D draw D K perpendicular to Q C, and D H perpendicular to M C. In Q C take C L, C R each a mean proportional between C Q, C K; and in M C take C F and C G each a mean proportional between C M, C H. Then will R L, G F be the axes of the ellipſe, or hyperbola, propoſed to be deſcribed, as is evident from Cor. 2. Prop. IX. (and 31. iii.) and Prop. VII. Book II. Conſequently the foci may be found, and the deſcriptions of the curves may be completed, as in the 5th and 6th Articles.

Fig. 116. 9. The ſtraight line G E being given in poſition and magnitude, and the ſtraight line A B biſecting it in B, let it be required to deſcribe a parabola, of which A B ſhall be a diameter, and G E a double ordinate to it. Let the ſtraight line P be a third proportional to A B, B G, and produce B A to Y, ſo that A Y may be a fourth part of P. Through Y draw D X at right angles to Y B, and through A draw A N parallel to G E. Make the angle N A F equal to the angle N A Y, and make A F equal to A Y. A parabola deſcribed with the focus F and the directrix D X, as in the 7th Article, will be the ſection required, as is evident from Prop. II. IX. and Cor. 2. and 3. Prop. XI.

10. The

and P L an ordinate to it being given, let it be required to find the diameter conjugate to A B. Fig. 111. 112.

Let the straight line M be a mean proportional (13. vi.) between the abscisses A F, F B; and, C being the center of the section, let M be to P L as C B to C D a straight line parallel to P L. Then will C D be half the conjugate diameter required. For, by hypothesis, (and 22. vi.) $M^2 : PL^2 :: CB^2 : CD^2$. But (17. vi.) $M^2 = AL \times LB$, and therefore $AL \times LB : PL^2 :: CB^2 : CD^2$. Consequently C D is the semiconjugate diameter to A B, by Prop. IV. and V. and Def. VIII. Book II.

Fig. 111.

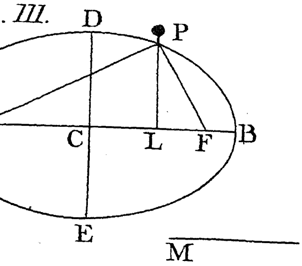


Fig. 110.

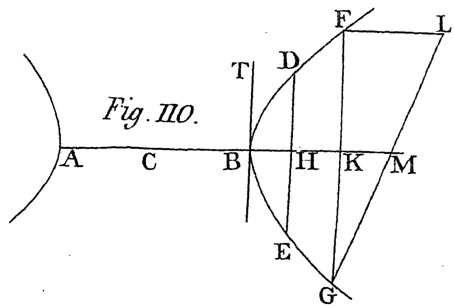


Fig. 109.

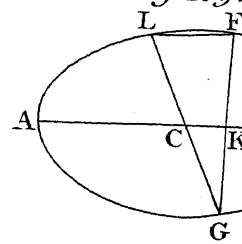


Fig. 112.

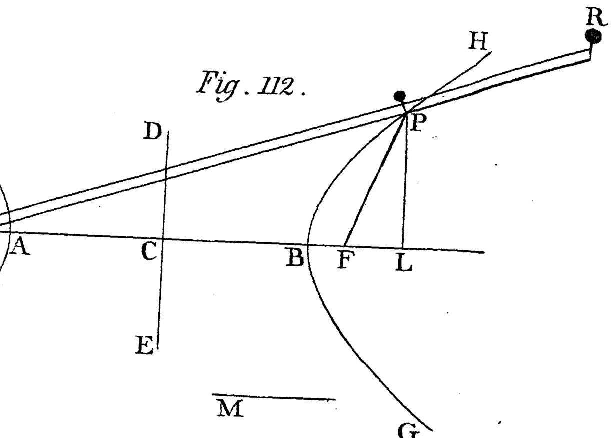


Fig. 114.

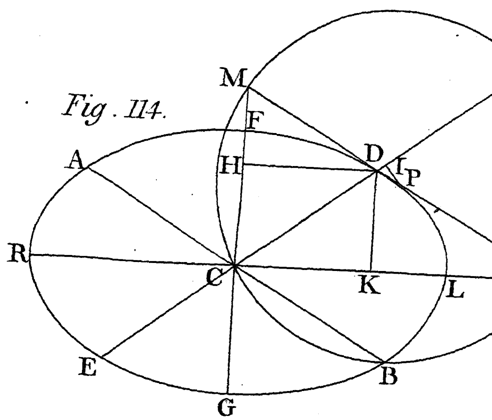


Fig. 113.

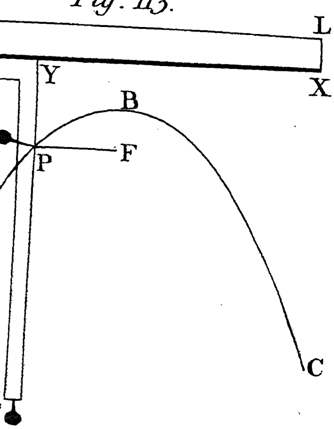


Fig. 116.

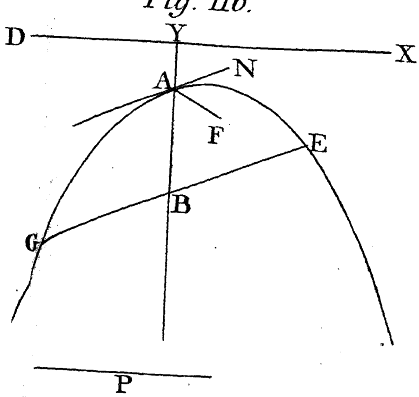
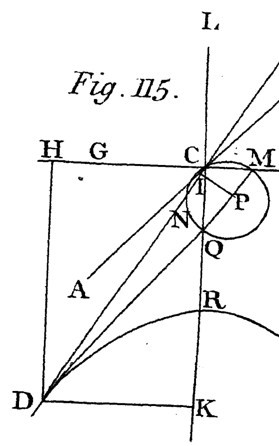


Fig. 115.



LEMMAS

FOR

THE FOURTH BOOK

OF

CONIC SECTIONS.

LEMMA IX.*

Let it be required to draw a straight line to touch two given circles A L B, D E M.

Let C be the center of A L B, and F the center of D E M, and draw C F. Let C G be the excess of the radius of A L B above the radius of D E M; and with C as a center, and C G as a distance, describe the circle H G. From the point F draw F H (17. iii.) touching the circle H G in H. Draw C H, and let it meet the circumference of A L B in A; and draw F D parallel to C A, and let it meet the circumference of D E M in D. Draw A D, and it will touch the given circles. For by the construction A H, D F are equal and parallel, and (16. iii.) A H F is a right angle. Consequently (33. i.) A D F H is a parallelogram, and (Cor. 46. i.) the Fig. 128.

* The numbering of the Lemmas is continued from those prefixed to the first Book. This manner of numbering them was found most convenient for reference.

the angles at A and D are right ones, and therefore (16. iii.) A D touches the circles.

If it be required to draw a ſtraight line to touch the circles and cut C F, let C F be ſo divided in K that C K may be to K F as the radius of the circle A L B to the radius of the circle D E M. Draw K L (17. iii.) to touch the circle A L B in L. Draw C L and F M parallel to it, and let F M meet L K in M; and L K will touch the circle D E M in M. For (15. and 29. i.) the triangles C K L, F K M are equiangular, and therefore (4. vi.) C K : K F : : C L : F M. Conſequently, by the conſtruction, the point M is in the circumference of the circle D E M, and as the angles F M K, C L K are equal, and the angle C L K a right one, L M (16. iii.) touches the circle D E M in M.

LEMMA X.

If a magnitude A *be to a magnitude* B *as a magnitude* C *to a magnitude* D; *then* A *will be to* B *as the difference of the antecedents* A, C *to the difference of the conſequents* B, D.

For let A be greater than C, and conſequently (14. v.) B greater than D. Then, by alternation, A : C : : B : D, and (17. v.) A — C : C : : B — D : D; and again, by alternation, C : D : : A — C : B — D. But, by hypotheſis, A : B : : C : D, and therefore (11. v.) A : B : : A — C : B — D.

DEFINITIONS.

I.

Fig. 129. 130. If the ſtraight line A D be ſo divided in the points B, C, that the whole line A D be to one of the extreme parts A B, as the other extreme part D C to the middle part B C, the ſtraight line A D is ſaid to be *Harmonically*

monically divided; and the points A, B, C, D are called the *Points of harmonical division*, or *Harmonical Points*.

Cor. 1. It is evident that each extreme part is greater than the middle part.

Cor. 2. If the two extreme points A, D, and B one of the middle points of an harmonical divifion be given, the other point C may be found (10. vi.) by dividing the fegment B D in C, fo that the part C D may be to B C as A D to A B. It is evident that no other point befides C can be found, which can be a fourth point of this divifion.

Cor. 3. The two middle points B, C, and A one of the extreme points of an harmonical divifion being given, the other point D may be found. For from the point A draw the ftraight line A E, and as A B to B C, fo let A E be to the fegment E F, taken towards A. Draw F C, and let E D drawn parallel to F C meet A C produced in D. Then (2. vi.) A D : D C : : A E : E F; and therefore by the conftruction A D : D C : : A B : B C, and confequently D is the other point of the divifion. It is evident that no other point befides D can be found, which can be a fourth point of this divifion. For, by converfion, as the excefs of A B above B C to B C, fo is A C to C D. Fig. 130.

II.

If the ftraight line *a d* be divided harmonically in the points *a*, *b*, *c*, *d*, and four ftraight lines *a* E, *b* E, *c* E, *d* E, any way produced through the points of divifion, be parallel or meet one another in the point E; thefe four ftraight lines are called *Harmonicals*. Fig. 129. 130.

Cor. Every thing remaining as above, any ftraight line A D, parallel to *a d*, and meeting the harmonicals in A, B, C, D, will be harmonically divided in thefe points. For, (29. and 15. i.) on account of the equiangular triangles,

angles, $ad : dE :: AD : DE$, and $dE : dc :: DE : DC$. Consequently,

$$ad : dE : dc$$
$$AD : DE : DC,$$

and (22. v.) $ad : dc :: AD : DC$. Again $ab : bE :: AB : BE$, and $bE : bc :: BE : BC$; and therefore

$$ab : bE : bc$$
$$AB : BE : BC.$$

Consequently (22. v.) $ab : bc :: AB : BC$; and therefore (11. v.) $AD : DC :: AB : BC$, as, by hypothesis, $ad : dc :: ab : bc$.

LEMMA XI.

Fig. 129. 130. *The rest remaining as in the first and second Definitions and their Corollaries, if a straight line* GH *parallel to any one of the harmonicals* aE, bE, cE, dE, *meet the other three, it will be bisected in the middle point of concourse. And, on the contrary, if four straight lines* ED, EC, EB, EA *meet one another in* E, *and if the straight line* GH *parallel to any one of them, and meeting the other three, be bisected in the middle point of concourse, the straight lines* ED, EC, EB, EA *will be harmonicals.*

Fig. 130. Part I. First, let the straight line GH be parallel to the harmonical dE, and meet aE, bE, cE in G, B, H; GH is bisected in the middle point B. For through B draw the straight line AD parallel to ad, and meeting the harmonicals aE, cE, dE in A, C, D. Then the straight line AD is harmonically divided, by the Cor. to the second Definition, and therefore $DA : AB :: DC : CB$. But (4. vi.) $DA : AB :: DE : BG$; and $DC : CB :: DE : BH$. Consequently (11. v.) $DE : BG :: DE : BH$, and therefore (14. v.) GB, BH are equal.

Se-

Secondly, let the ſtraight line G H be parallel to the harmonical c E, and meet a E, b E, d E in A, H, G; it will be biſected in the middle point A. For through A draw the ſtraight line A D parallel to $a\,d$, and meeting the harmonicals b E, c E, d E in B, C, D. Then, by the Cor. to the ſecond Definition, A D : D C : : A B : B C. But (4. vi.) A D : D C : : G A : E C; and A B : B C : : A H : E C. Conſequently (11. v.) G A : E C : : A H : E C, and therefore (14. v.) G A, A H are equal. Fig. 129.

Part II. Firſt, let the ſtraight line G H be parallel to A D, and meet the ſtraight lines E C, E B, E A in H, B, G, and let it be biſected in the middle point B. Through B draw the ſtraight line A D, and let it meet the ſtraight lines E A, E C, E D in A, C, D. Then (4. vi.) D E : B G : : D A : A B, and D E : B H : : D C : C B. Conſequently (7. and 11. v.) D A : A B : : D C : C B. Fig. 130.

Secondly, let the ſtraight line G H be parallel to E C, and meet the ſtraight lines E D, E B, E A in the points G, H, A, and let it be biſected in the middle point A. Through A draw the ſtraight line A D, meeting the ſtraight lines E B, E C, E D in B, C, D. Then (4. vi.) A G : E C : : A D : D C; and A H : E C : : A B : B C. Conſequently (7. and 11. v.) A D : D C : : A B : B C. Fig. 129.

LEMMA XII.

If four harmonicals meet any ſtraight line, the ſtraight line will be harmonically divided in the points of concourſe.

If the harmonicals are parallel to one another, this is evident (from 10. vi.), but if not, let a E, b E, c E, d E be four harmonicals, and let them meet any ſtraight line A D in A, B, C, D. Through B draw the ſtraight line G H parallel to d D, and let it meet the ſtraight line E A in G, and E C in H. Then, by the preceding Fig. 130.

Lemma,

Lemma, G H is bisected in B; and (4. vi.) D A : A B :: D E : B G or B H; and D E : B H :: D C : C B. Consequently (11. v.) D A : A B :: D C : C B.

A GEO-

A GEOMETRICAL TREATISE OF *CONIC SECTIONS.*

BOOK IV.

Of ſimilar Sections, general Properties, Circles having the ſame curvature with the Sections in given points, and of ſtraight lines cut harmonically by the Sections. This Book alſo contains Problems uſeful in the Theory of Aſtronomy, and Methods of finding two mean proportionals and of triſecting an angle, by means of the Sections.

DEFINITIONS.

I.

TWO ſegments of conic ſections are called *Similar Segments*, if a rectilineal figure can be inſcribed in one of them ſimilar to any rectilineal figure inſcribed in the other.

II.

Two conic ſections are called *Similar Sections*, if a rectilineal figure can be inſcribed in one of them ſimilar

BOOK IV. lar to any rectilineal figure inscribed in the other: and two conic sections are also called *Similar*, if a segment can be taken of one of them similar to a segment of the other.

III.

If a straight line touch two conic sections in the same point, the two sections are said to *touch one another* in the same point.

IV.

If a circle so touch a conic section in any point, that no other circle, touching it in the same point, can pass between it and the section, on either side of the point of contact, it is said to have the *same Curvature with the Section in the point of contact*; or it is said to be the *Osculating Circle for that point.*

PROP. I.

Any two parabolas are similar to one another; and the similar rectilineal figures inscribed in them are to one another as the squares of the parameters of the axes.

Fig. 117. 118. Let A B C, *a b c* be two parabolas, of which B E, *b e* are the axes, and P, *p* their parameters; and let A B C D be any rectilineal figure inscribed in the one parabola; a rectilineal figure similar to A B C D may be inscribed in the other, and the similar rectilineal figures inscribed in them are to one another as P^2 to p^2.

For draw from *b*, the vertex, the straight line *b a* to the curve, so that the angle *e b a* may be equal to the angle E B A. Draw *b c* to the curve, so that the angle *e b c* may be equal to the angle E B C. Draw B D, and draw *b d* to the curve, so that the angle *e b d* may be equal to the angle E B D; and draw *a d*. Let A E be an ordinate to the axis B E, and let *a e* be an ordinate to the axis *b e*. Then, as the angles at E and *e* are right

right angles, the triangles A B E, *a b e* are equiangular, and (4. vi.) B E : E A : : *b e* : *e a*. But, by the fixth Definition Book III. B E : E A : : E A : P; and *b e* : *e a* : : *e a* : *p*. We have therefore the two following ranks of magnitudes, in which the magnitudes taken two and two in the fame order have the fame ratio to one another;

B E : E A : P

b e : *e a* : *p*

and confequently (22. v.) B E : P : : *b e* : *p*; or by alternation B E : *b e* : : P : *p*. But (4. vi. and altern.) B E : *b e* : : B A : *b a*, and therefore (11. v.) B A : *b a* : : P : *p*. In the fame way it may be proved, that B C : *b c* : : P : *p*, and that B D : *b d* : : P : *p*; and therefore (11. v.) B C : *b c* : : B D : *b d*, and by alternation B C : B D : : *b c* : *b d*. But as the angles E B D, *e b d* are equal, and the angle E B C equal to the angle *e b c*, the angles D B C, *d b c* are equal. Confequently (6. vi.) the triangles C B D, *c b d* are equiangular, and (4. vi. and altern.) C D : *c d* : : B C : *b c*, or P to *p*. In the fame way it may be proved that A D : *a d* : : P : *p*; and therefore the rectilineal figure *a b c d* is fimilar to the rectilineal figure A B C D. The parabolas A B C, *a b c* are therefore fimilar according to the firft and fecond Definitions, and it is evident (Cor. 2. 20. vi.) that the rectilineal figures A B C D, *a b c d* are to one another as the fquares of their homologous fides, or (11. v.) as P^2 to p^2.

Cor. 1. It is evident from the above that fimilar rectilineal figures may be infcribed in the fimilar parabolic fegments A B C D, *a b c d*, of which the homologous fides will be to one another as P to *p*, and which will be deficient from the parabolic fegments by fpaces lefs than any given. The fimilar parabolic fegments themfelves will therefore be to one another as P^2 to p^2.

M *Cor.*

BOOK IV.

Cor. 2. In two parabolas the parameters of diameters, which contain equal angles with their ordinates, are to one another as the parameters of the axes. For, the reſt remaining as above, let A G, touching the parabola A B C in A, meet the axis B E in G; and let $a g$, touching the parabola $a b c$ in a, meet the axis $b e$ in g. Let F be the focus of A B C, and f the focus of $a b c$; and draw A F, $a f$. Then, as B F is one fourth of P, and $b f$ one fourth of p, by the above, (and 15. v.) A B : B F : : $a b : b f$; and therefore (6. vi.) the triangles A B F, $a b f$ are equiangular, and A F : B F : : $a f : b f$; or, by Cor. 2. Prop. XI. Book III. (and 15. v.) the parameter of the diameter paſſing through A is to P as the parameter of the diameter paſſing through a to p. Again, by Cor. Prop. IX. Book III. the triangles A F G, $a f g$ are iſoſceles, and, as above, the angles A F G, $a f g$ are equal. The angles A G F, $a g f$ are therefore equal; and, as the diameters of a parabola are parallel, the angle A G F is equal to the angle which the diameter paſſing through A contains with its ordinates, and the angle $a g f$ equal to the angle which the diameter paſſing through a contains with its ordinates, by Prop. II. Book III.

PROP. II.

Two ellipſes, or two hyperbolas, are ſimilar to one another, if two conjugate diameters in the one be proportional to two conjugate diameters in the other, and the firſt two and the other two contain equal angles. On the contrary, if two ellipſes, or two hyperbolas, be ſimilar to one another, two conjugate diameters in the one will be proportional to two conjugate diameters in the other, provided the firſt two and the other two contain equal angles.

Let

Fig. 119. 120. 121. 122.

Let B H, *b h* be two ellipses, or two hyperbolas, and in the one let A B a diameter be to D E its conjugate as, in the other, *a b* a diameter is to *d e* its conjugate, and, C, *c* being the centers, let the angle D C B be equal to the angle *d c b*; then the ellipse B H is similar to the ellipse *b h*, and the hyperbola B H is similar to the hyperbola *b h*. On the contrary, if the ellipse or hyperbola B H be similar to the ellipse or hyperbola *b h*, and if the angle D C B contained by A B, D E, two conjugate diameters in the one, be equal to the angle *d c b* contained by *a b*, *d e*, two conjugate diameters in the other; then A B is to D E as *a b* to *d e*.

Part I. Let B H M L K be any rectilineal figure inscribed in the ellipse or hyperbola B H, and from C the center draw the straight lines C H, C M, C L, C K. Again, at *c* the center of the ellipse or hyperbola *b h*, make the angles *b c h*, *h c m*, *m c l*, *l c k*, equal to the angles B C H, H C M, M C L, L C K, each to each; and let the points *h*, *m*, *l*, *k* be in the curve of the section. Then the straight lines *b h*, *h m*, *m l*, *l k*, *k b* being drawn, the rectilineal figure *b h m l k* inscribed in the ellipse *b h* is similar to B H M L K inscribed in the ellipse B H; and the rectilineal figure *b h m l k* inscribed in the hyperbola *b h* is similar to B H M L K inscribed in the hyperbola B H. For draw H G an ordinate to A B, and *h g* an ordinate to *a b*. Then, as the angles D C G, *d c g* are equal, and as, by Cor. 2. Prop. IV. Book II. H G is parallel to D C, and *h g* parallel to *d c*, the angles (29. I.) at G and *g* are equal. The angles H C G, *h c g* are also equal, by construction, and therefore the triangles H C G, *h c g* are equiangular. Consequently (4. vi.) $GH : CG :: gh : cg$; and $GH^2 : CG^2 :: gh^2 : cg^2$, and (16. v.) $GH^2 : gh^2 :: CG^2 : cg^2$. But, by hypothesis, (and 15. v.) $CB : CD :: cb : cd$, and therefore $CB^2 : CD^2 :: cb^2 : cd^2$; and,

 by

BOOK IV.

by Prop. V. Book II. $CB^2 : CD^2 :: AG \times GB : GH^2$; and $cb^2 : cd^2 :: ag \times gb : gb^2$. Hence (11. v.) $AG \times GB : GH^2 :: ag \times gb : gb^2$; and (16. v.) $AG \times GB : ag \times gb :: GH^2 : gb^2$. Consequently (11. v.) $AG \times GB : ag \times gb :: CG^2 : cg^2$; and therefore (16. v.) $AG \times GB : CG^2 :: ag \times gb : cg^2$, and (by 5. and 6. ii. and 17. and 18. v.) $CB^2 : CG^2 :: cb^2 : cg^2$. We have therefore (22. vi.) $CB : CG :: cb : cg$; and (16. v.) $CB : cb :: CG : cg$. Again, by the similar triangles, $CG : cg :: CH : cb$; and therefore (11. v.) $CB : cb :: CH : cb$, and by alternation $CB : CH :: cb : cb$. Consequently (6. vi.) the triangles BCH, bcb are similar. In the same way it may be proved that $CB : cb :: CM : cm$; and therefore (11.v.) that $CH : cb :: CM : cm$. Consequently, by alternation, (and 6. vi.) the triangles MCH, mcb are similar; and in the same way it may be proved, that the triangle lcm is similar to the triangle LCM, the triangle lck to the triangle LCK, and the triangle kcb to the triangle KCB. The rectilineal figures BHMLK $bbmlk$ (20. vi.) are therefore similar, and the sections BH, bb are therefore similar, according to the first and second Definitions.

Part II. If ab be not to de as AB to DE, let AB be to DE as ab to a straight line greater or less than de; and with this straight line as a conjugate diameter, and AB as a transverse diameter, suppose an ellipse or hyperbola to be described. Then this ellipse or hyperbola will fall without or within the section bb of the same name, and, by the preceding part, a rectilineal figure may be inscribed in the section, at present supposed to be described, similar to the rectilineal figure BHMLK. But as, by hypothesis, the sections BH, bb are similar, a rectilineal figure may be inscribed in the section bb similar to the rectilineal figure BHMLK, ac-

according to the first and second Definitions. Let this inscribed figure be *b b m l k*. Then (21. vi.) in the section, having *a b* for its transverse diameter, and falling either without or within the section *b b*, a rectilineal figure may be inscribed similar to *b b m l k*; which (from Def. i. vi. and 21. i.) is evidently absurd. Consequently, the sections B H, *b b* being similar, and the angles D C B, *d c b* equal, A B is to D E as *a b* to *d e*.

Cor. 1. In similar ellipses, or similar hyperbolas, diameters which contain equal angles with the axes are to one another as the axes. For if A B, *a b* be the transverse, and D E, *d e* the conjugate axes, then the angles B C H, *b c b* being equal, C B is to C H as *c b* to *c h*, as was above demonstrated.

Cor. 2. From the above (and Cor. 2. 20. vi.) it is evident that similar rectilineal figures may be inscribed in similar ellipses, or in similar hyperbolic segments, which shall be to one another as the squares of the transverse, or as the squares of the conjugate axes.

Cor. 3. Similar ellipses, and similar hyperbolic segments, are to one another as the squares of their transverse, or as the squares of their conjugate axes. This is evident from the preceding Cor. (and 2. xii.) as a rectilineal figure may be inscribed in an ellipse, or in a hyperbolic segment, which shall be deficient from the ellipse, or hyperbolic segment, by a space less than any given space.

Cor. 4. The angles contained by the asymptotes of similar hyperbolas are equal to one another; and if the angles contained by the asymptotes of two hyperbolas be equal, the hyperbolas will be similar. For let A B, D E be the axes, C the center, and C S an asymptote of the hyperbola B H; and let *a b*, *d e* be the axes, *c* the center, and *c s* an asymptote of the hyperbola *b h*. Let B S touch the hyperbola B H in the vertex B, and meet the

BOOK IV.

the asymptote in S; and let *b s* touch the hyperbola *b b* in the vertex *b*, and meet the asymptote in *s*. Then the angle C B S is equal to the angle *c b s*, as each is a right one, and, by Cor. 3. Prop. XV. Book III. B S is equal to C E, and *b s* is equal to *c e*; and therefore, if the hyperbolas be similar, by the second part of this Prop. (and 15. v.) C B : B S : : *c b* : *b s*. Consequently (6. vi.) the angle B C S is equal to the angle *b c s*. On the contrary, if the angle B C S be equal to the angle *b c s*, then (4. vi.) C B : B S : : *c b* : *b s*; and therefore, by the first part of this Prop. the hyperbolas B H, *b b* are similar.

PROP. III.

If two ellipses, or two hyperbolas, be similar, the transverse axis in the one will be to the distance between the foci as the transverse axis in the other to the distance between the foci. On the contrary, two ellipses, or two hyperbolas, will be similar, if the transverse axis in the one be to the distance between the foci as the transverse axis in the other to the distance between the foci.

Fig. 119. 120. 121. 122.

Part I. Let B H, *b b* be two similar ellipses, and B H, *b b* be two similar hyperbolas. Let A B, D E be the transverse and conjugate axes, C the center, and F, O the foci of the one; and let *a b*, *d e* be the transverse and conjugate axes, *c* the center, and *f*, *o* the foci of the other; then A B is to F O as *a b* is to *f o*.

For in the ellipses draw E O, *e o*, but in the hyperbolas draw E A, *e a*. Then, by Cor. 2. and Cor. 3. Def. XI. Book II. in the ellipses E O is equal to C A. and *e o* is equal to *c a*; but in the hyperbolas E A is equal to C O, and *e a* is equal to *c o*. Hence, as the section B H is similar to the section *b b*, by the second part of Prop. II. in the ellipses E O : E C : : *e o* : *e c*; but

but in the hyperbolas C A : C E : : *c a* : *c e*. Consequently, as the angles at C and *c* are equal, being right angles, the triangles (7. vi.) E C O, *e c o* in the ellipses are equiangular; and in the hyperbolas the triangles (6. vi.) A C E, *a c e* are equiangular. Hence in the ellipses E O, or its equal C A, is to C O as *e o*, or its equal *c a*, is to *c o*; but in the hyperbolas C A is to A E, or its equal C O, as *c a* is to *a e*, or its equal *c o*.

Part II. On the contrary, the rest remaining as above, if it be allowed, either in the ellipses or hyperbolas, that C A is to C O as *c a* to *c o*, then it may be proved, as above, (by 7. vi.) that in the ellipses the triangles E C O, *e c o* are equiangular, but in the hyperbolas that the triangles A C E, *a c e* are equiangular. Consequently in the ellipses E O, or its equal C A, is to C E as *e o*, or its equal *c a*, to *c e*; and in the hyperbolas C A is to C E as *c a* to *c e*. Hence this part of the Cor. is evident from the first part of Prop. II.

Cor. If B H, *b h* be two ellipses, or two hyperbolas, of which the foci are O, F in the one, and *o*, *f* in the other, and if the triangles M O F, *m o f* be equiangular; then if the sections be similar, and the point M be in the curve of B H, the point *m* will be in the curve of *b h*. For (4. vi.) O M : M F : : *om* : *mf*; and therefore (18. v.) in the ellipses O M + M F : M F : : *om* + *mf* : *mf*; and (17. v.) in the hyperbolas O M — M F : M F : : *om* — *mf* : *mf*. Again, (4. vi.) either in the ellipses or hyperbolas, M F : F O : : *mf* : *fo*. Consequently, in the ellipses,

$$OM + MF : MF : FO$$
$$om + mf : mf : fo;$$

and therefore (22. v.) O M + M F : F O : : *om* + *mf* : *fo*. Also, in the hyperbolas,

$$OM - MF : MF : FO$$
$$om - mf : mf : fo;$$

 and

BOOK IV. and therefore (22. v.) O M − M F : F O : : $om - mf$: fo. But, the reft remaining as in the Propofition, by Prop. XIII. Book II. O M + M F in the ellipfe, but O M − M F in the hyperbola, is equal to A B, and as the fections are fimilar, by this Prop. A B : F O : : ab : fo. Confequently (11. v.) in the ellipfe $ab : fo$: : $om + mf : fo$; and in the hyperbola $ab : fo : : om - mf : fo$. In the ellipfe, therefore, (14. v.) $om + mf$ is equal to ab, and in the hyperbola $om - mf$ is equal to ab. Confequently the point m is in the curve of the fection bb, by Cor. 1. Prop. XIV. Book II.

SCHOLIUM.

Having explained methods for defcribing conic fections, and demonftrated the principal properties of fimilar fections, it may be proper, in this place, to give an explanation of fome paffages in the fourth fection of the firft Book of the Principia. The reader, who thinks fuch explanations neceffary, is fuppofed to have that juftly celebrated work before him whilft he perufes this Scholium, as he is here referred to the figures which it contains. In the following explanations the Lemmas and Propofitions of the above-mentioned Section of the Principia are printed in capital letters, to diftinguifh them from fuch parts of this treatife as are referred to.

LEMMA XV. This is fully explained in Cor. 2. Prop. XV. Book II.

PROPOSITION XVIII. This is evident from Prop. XIII. and Cor. 2. Prop. XV. Book II.

PROPOSITION XIX. The reafons for the defcriptions here delivered are eafily deduced from Cor. 3. Prop. X. and Cor. 3. Prop. VIII. Book III. and Lemma IX. and it is evident that F I will be the directrix of the parabola.

PROPO-

PROPOSITION XX. An ellipfe or hyperbola is faid to be given in fpecies when it is fimilar to a given ellipfe or hyperbola, or when the ratio of the axes to one another is given. When the ratio of the axes to one another is given, the ratio of the principal or tranfverfe axis to the diftance of the foci is alfo given, or is conftant, according to Prop. III.

The remaining part of Cafe 1. PROP. XX. is evident from Cor. 1. Prop. VIII. Book III. and Lemma IX.

Cafe 2. That a ftraight line bifecting V v at right angles will pafs through the other focus, of the ellipfe or hyperbola, has been proved in Cor. 3. Prop. XV. Book II. and that the circumference of a circle defcribed upon K k as a diameter will pafs through the fame focus has been proved in the 4th Cor. to the fame Propofition. The whole of Cafe 2. is therefore evident from the Corollaries already mentioned, and from Prop. XIII. and XV. Book II.

Cafe 3. That a circle defcribed upon K k as a diameter will pafs through the other focus is evident from Cor. 4. Prop. XV. Book II. and that V R will pafs through the fame focus is evident from Prop. XV. Book II.

Cafe 4. That $v\,b$ is equal to $a\,b$ is evident from Sir Ifaac Newton's demonftration. Recurring to proportions previoufly ftated S H : $s\,b$: : S P : $s\,p$, and therefore S H : S P : : $s\,b$: $s\,p$; and as the angles P S H, $p\,s\,b$ are equal, the triangles P S H, $p\,s\,b$ are fimilar. Again $s\,v$: $s\,p$: : $s\,b$: $s\,q$; and on account of the fimilar triangles V S P, $b\,s\,q$, S V : S P : : $b\,s$: $s\,q$. Confequently (11. v.) S V : S P : : $s\,v$: $s\,p$, and by inverfion S P : S V : : $s\,p$: $s\,v$. From the above therefore

S H : S P : S V

$s\,b$: $s\,p$: $s\,v$,

and

BOOK IV.

and (22. v.) SH : SV :: *sh* : *sv*, and as the angles VSH, *vsh* are equal, the triangles (6. vi.) VSH, *vsh* are ſimilar. From hence, and Prop. III. and its Cor. the proceedings in this caſe are evidently juſt.

LEMMA XVI. Caſe 1. By Cor. 1. Prop. VIII. Book III. PR is the directrix of the hyperbola, of which NM is the tranſverſe axis, and A, B the foci, and therefore AP is given; and by the ſame Cor. ZR : AZ :: MN : AB. Let D denote the difference between AZ, CZ, and then D by hypotheſis is given. For the ſame reaſons AQ is given, SQ being the directrix to the other hyperbola; and, as above, AZ : ZS :: AC : D. Conſequently, by the fifth Lemma, (and 1. vi.) ZR : ZS :: MN × AC : AB × D; and as MN, AC, AB, and D are given, the ratio of ZR to ZS is given. Again, Radius : ſine $\angle$ ZTS :: TZ : ZS; and therefore by the above, and the fifth Lemma, (and 1. vi.) Radius × D : ſine $\angle$ ZTS × AC :: TZ : AZ. But TZ and TS being given in poſition, the ſine of the angle ZTS is given, and therefore the ratio of TZ to AZ is given. The ſtraight line TA is given, and alſo the angle ATZ, and AZ : TZ :: ſine $\angle$ ATZ : ſine $\angle$ TAZ; and therefore the angle TAZ is given. Conſequently the triangle ATZ is given. By introducing ſome additional ſtraight lines into the figure, the triangle might be aſcertained geometrically.

The other two caſes need no explanation; nor do the remaining parts of the Section, as the ſeveral caſes in PROP. XXI. may be ſolved by the preceding LEMMA. The SCHOLIUM is evident from Prop. VIII. Book III.

PROP. IV.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet a ſtraight line which cuts the ſection,

oppo-

opposite hyperbola, or opposite hyperbolas, and is parallel to the straight line joining the points of contact, the segments of the secant between the curve or curves and the tangents will be equal to one another: and if two straight lines touching a conic section meet a straight line which touches the section, or opposite hyperbola, and is parallel to the straight line joining the points of contact, the segments of the last mentioned tangent between the point of contact and the tangents which it meets will be equal to one another. BOOK IV.

If the two tangents be parallel, this Prop. is the same as the Cor. to Prop. I. Book II. and this case has been already demonstrated. Let the two straight lines therefore A E, D E touch the conic section A D, or the opposite hyperbolas A, D, in the points A, D, and meet one another in E, and, first, let them meet in the points H, K the straight line H K, which is parallel to A D, and cuts the curve of the section, the opposite hyperbola, or the curves of the opposite hyperbolas, in the points F, G; the segments H F, K G are equal to one another. Fig. 131. 132. 133. 134.

For let the straight line E M bisect A D in M, and meet the straight line H K in N; and then, by Cor. 1. to Prop. VI. Book III. E M is a diameter, and therefore, as A D, F G are parallel, F G is bisected in N, by Cor. 1. to Prop. II. Book III. But (29. i. and 4. vi.) E M : E N : : A M : H N, and E M : E N : : D M : K N. Consequently (11. v.) A M : H N : : D M : K N, and therefore (14. v.) H N is equal to K N, and H F equal to K G.

The rest remaining as above, let the tangents A E, D E now meet the straight line L I in the points L and I, and let L I be parallel to A D, and touch the section, Fig. 131. 132. 134.

or

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or oppofite hyperbola, in the point B; the fegments L B, B I are equal.

For as L I, A D are parallel, by Prop. II. Book III. a diameter paffing through B will bifect A D, and, by Prop. VI. Book III. it will pafs through E. The diameter E M therefore paffes through B the point of contact, and (4. vi.) E M : E B : : A M : L B. Alfo E M : E B : : D M : I B, and therefore (11. v.) A M : L B : : D M : I B. Confequently (14. v.) L B is equal to I B.

Fig. 131. 132. 133. 134. *Cor.* 1. If two parallel ftraight lines as A D, G F cut a conic fection, or oppofite hyperbolas, in A, D and G, F, and ftraight lines A G, D F joining their extremities meet in O, Q the ftraight line O Q, which cuts the fection, or oppofite hyperbolas, in P, R, and is parallel to A D, G F; the fegments O R, Q P are equal. For let the tangents A E, D E meet O Q in T and S; and then (4. vi.) A H : A T : : H G : T O, and D K : D S : : K F : S Q. But, on account of the parallels, A H : A T : : D K : D S, and therefore H G : T O : : K F : S Q; and as, by this Prop. H G is equal to K F, and T R to S P, it is evident (14. v.) that O R is equal to Q P.

Fig. 131. 132. 134. *Cor.* 2. The reft remaining as above, let A G meet the tangent L I in V, and let D F meet it in Y, and the fegments V B, Y B will be equal. For it may be demonftrated, as in the preceding Cor. that L V, I Y are equal; and confequently, as L B, I B are equal, the fegment V B is equal to the fegment Y B.

PROP. V.

If a trapezium be infcribed in a conic fection, or oppofite hyperbolas, and its fides be indefinitely produced, and if from any point in the curve two ftraight lines be drawn parallel to two adjacent fides of the trapezium and meet the oppofite fides; the rectangles under the fegments of thefe ftraight

straight lines, between the point in the curve and the opposite sides of the trapezium, will be to one another as the squares of the semidiameters parallel to them, or the squares of the tangents parallel to them, and meeting one another. BOOK IV.

Let A B D C be a trapezium inscribed in a conic section, or opposite hyperbolas, and let its sides be indefinitely produced, and from any point E in the curve let the straight lines E N, E H be drawn parallel to the adjacent sides A B, A C, and let E N meet the opposite sides A C, B D in Q and N, and E H meet the opposite sides A B, C D in R and H; the rectangles Q E N, H E R are to one another as the squares of the semidiameters parallel to E N, E H, or the squares of the tangents parallel to them and meeting one another. Fig. 123. 124.

For if the opposite sides A C, B D be not parallel, let the straight lines B *d*, D F be drawn parallel to A C or H R; and let B *d* meet the curve again in *d*, and let it meet E N in *n*. Let D F meet the curve again in F, and the straight line A B in G. Draw the straight line C *d*, and let it meet the straight line H R in *b*, and the straight line D F in S. Let the straight line H R meet the curve again in T. Then, by Cor. I. Prop. IV. S D, F G are equal, and *b* T is equal to E R, and consequently *b* E, T R are equal. The rectangle *b* E R is therefore equal to the rectangle T R E; and, on account of the parallelograms B E, A E, the rectangle Q E *n* is equal to the rectangle A R B. The rectangles T R E, A R B are therefore to one another as the rectangles *b* E R, Q E *n*; and consequently, by Prop. V. Book. II. and Prop. XIII. Book I. these rectangles are to one another as the squares of the semidiameters parallel to H R, E N, or as the squares of tangents parallel to H R, E N, and meeting one another. Again, on account of the

BOOK IV. the equiangular triangles H C b, D C S, H b : S D : : C b : C S; and therefore, on account of the equals and parallels, H b : F G : : E Q : A G. Alſo, on account of the equiangular triangles N B n, D B G, n B or (34. i.) E R : D G : : N n : G B; and therefore, by the fifth Lemma, H b × E R : D G × G F : : E Q × N n : A G × G B; and by alternation H b × E R : E Q × N n : : D G × G F : A G × G B. But, by Prop. V. Book II. and Prop. XIII. Book I. D G × G F and A G × G B are to one another as the ſquares of the ſemidiameters parallel to H R, E N, or as the ſquares of tangents parallel to H R, E N, and meeting one another. By the above therefore (and 11. v.) b E × E R : Q E × E n : : H b × E R : E Q × N n; and by alternation b E × E R : H b × E R : : Q E × E n : Q E × N n; and therefore (1. vi.) b E : H b : : E n : N n, and (17. and 18. v.) H E : H b : : E N : N n. Conſequently by alternation (and 1. vi.) H E × E R : E N × E Q : : H b × E R : N n × E Q; and therefore, by the above, (and 11. v.) H E × E R and E N × E Q are to one another as the ſquares of the ſemidiameters parallel to H R, E N, or as the ſquares of tangents parallel to H R, E N, and meeting one another.

Fig. 125. If the ſtraight lines A B, C D cut one another like the diagonals of a trapezium, as in Fig. 125. and the reſt be as expreſſed above in the particular enunciation; then it may be proved, as above, that Q E × E N and H E × E R are to one another as the ſquares of the ſemidiamers parallel to E N, E H, or the ſquares of the tangents parallel to them, and meeting one another.

It is evident that the method of demonſtration and concluſion will be the ſame, if one of the ſtraight lines H R, B d, D F, or even two of them, be tangents.

Fig. 123. 124. 125. *Cor.* 1. If the points A, B, C, D remain fixed, and the point E with the ſtraight lines E N, E H, always parallel

Q E N, H E R will be to one another in the ſame ratio.

Cor. 2. The points A, B, C, E remaining fixed, if the point D, the interſection of the ſtraight lines B D, C D, be carried round the ſection, or oppoſite hyperbola, in every ſituation of D the ſegments E H, E N will be to one another in the ſame ratio. For the rectangles H E R, Q E N in every ſituation of D will be to one another in the ſame ratio, and therefore as E R, Q E remain fixed, the Cor. is (1. vi.) evident.

Cor. 3. The reſt remaining as above, draw B C, and let it meet E H in L. In the ſtraight line E Q, and on the ſame ſide of E H with the point N, let the point T be taken, and let H E be to E N as L E to E T; and then B T being drawn, it will touch the ſection. For, if it be poſſible, let it meet the curve again in V; and V C being drawn, let it meet E H in X. Then, by Cor. 2. X E : E T : : H E : E N, and therefore (11. v.) L E : E T : : X E : E T, and L E, X E are equal: which is abſurd. The ſtraight line B T therefore touches the ſection. Fig. 126.

Cor. 4. Hence, the reſt remaining, if the ſtraight line B T touch the ſection in B, and meet the ſtraight line E N in T, H E will be to E N as L E to E T.

Cor. 5. If the ſtraight lines A R, C H touch the ſection in A, C, and from E, a point in the curve, E Q be drawn parallel to A R, and meet A C, the line joining the points of contact, in Q, and H E R be drawn parallel to A C, and meet the tangents in R and H; then the rectangle H E R and the ſquare of E Q will be to one another as the ſquares of the ſemidiameters parallel to H R, Q E, or as the ſquares of the tangents parallel to H R, Q E, meeting one another. For if H R meet the curve again in T, then, by Prop. IV. H T, E R are equal, and the rectangles H E R, T R E are equal: Fig. 127.

BOOK IV.

equal: and (34. i.) as E Q is equal to A R, the Cor. is evident from Prop. V. Book II. and Prop. XIII. Book I.

PROP. VI.

If a trapezium be inscribed in a conic section, or opposite hyperbolas, and from any point in the curve a straight line be drawn in a given angle to each of the sides, or the sides produced, the rectangle under the lines drawn to two opposite sides will be to the rectangle under the lines drawn to the two other opposite sides in a given ratio.

Fig. 135. 136. Let A B D C be a trapezium inscribed in the conic section E A B D C, or in the opposite hyperbolas A D C, *e* B, and from any point E in the curve let the straight lines E L, E M, E O, E K be drawn in given angles to the sides A B, C D, B D, A C, each to each; the rectangle under E L, E M, drawn to the opposite sides A B, C D, is to the rectangle under E O, E K, drawn to the other two opposite sides B D, A C, in a given ratio.

For from any other point *e* in the curve of the section, or in the curve of either of the opposite hyperbolas, let the straight lines *e l*, *e m*, *e o*, *e k* be drawn parallel to E L, E M, E O, E K, each to each, and to the same side of the trapezium each to each. Through the point E let the straight lines Q N, H R be drawn parallel to the adjacent sides A B, A C, and meeting the sides A C, B D, A B, D C, in the points Q, N and R, H. Through the point *e* draw the straight lines *q n*, *h r* parallel to the straight lines Q N, H R, or to the sides A B, A C, and meeting the sides A C, B D, A B, D C in the points *q*, *n* and *r*, *h*. Then (29. i. and 4. vi.) E R : *e r* : : E L : *e l*; and E H : *e h* : : E M : *e m*. Consequently, by the fifth Lemma, E R $\times$ E H : *e r* $\times$ *e h* : :

E L.

EL × EM : $el \times em$. Again (29. i. and 4. vi.) EQ : eq : : EK : ek; and EN : en : : EO : eo, and therefore, as before, EQ × EN : $eq \times en$: : EK × EO : $ek \times eo$. But, by Cor. 1. Prop. V. ER × EH : $er \times eh$: : EQ × EN : $eq \times en$; and therefore (11. v.) EL × EM : $el \times em$: : EK × EO : $ek \times eo$. Consequently, by alternation, the rectangle under EL, EM is to the rectangle under EK, EO in the constant or given ratio of the rectangle under el, em to the rectangle under ek, eo.

Cor. If two straight lines AR, CH touch a conic section in A, C, and if from any point E in the curve straight lines EL, EM, EK be drawn in given angles to the tangents, and AC joining the points of contact, the rectangle under EL, EM and the square of EK will be to one another in a constant or given ratio. For take any other point e in the curve, and through E, e draw HR, hr parallel to AC, and let them meet the tangents in H, R and h, r; and draw EQ, eq parallel to AR, and let them meet AC in Q, q. Then by similar triangles, as above, ER × EH : $er \times eh$: : EL × EM : $el \times em$; and by Cor. 5. Prop. V. ER × EH : $er \times eh$: : EQ^2 : eq^2. But on account of the parallel lines, the triangles EKQ, ekq are similar, and EQ^2 : eq^2 : : EK^2 : ek^2. Consequently (11. v.) EL × EM : $el \times em$: : EK^2 : ek^2. Fig. 127.

PROP. VII.

The curve of a conic section cannot meet the curve of another conic section, or the curves of opposite hyperbolas, in more than four points.

For, if it be possible, let the curve of a conic section meet the curve of another conic section, or the curves of opposite hyperbolas, in the points A, B, D, C, E; Fig. 137.

 and

BOOK IV. and draw A B, B D, D C, C A. Let the ſtraight lines E N, E H be drawn parallel to A B, A C, and let them meet B D, D C in N and H. Let the ſtraight line B F be drawn, meeting the curves again in F, I, and the ſtraight line E N in n. Let the ſtraight lines F C, I C be drawn, and let them meet the ſtraight line E H in K and L. Then, by Cor. 2. Prop. V. H E : E N :: L E : E n, and H E : E N :: K E : E n. Conſequently (11. v.) L E : E n :: K E : E n, and (14. v.) L E, K E are equal: which is abſurd. The curve of the conic ſection, therefore, does not meet the curve of the other, or the curve of oppoſite hyperbolas, in five points.

Fig. 138. *Cor.* 1. If two conic ſections touch one another, they will not meet each other in three other points. For, if it be poſſible, let the two ſections have the common tangent in the point B, and meet one another in A, E, C, and of theſe let E be the intermediate point. Let B A, B C, C A be drawn; and let E T, E H be drawn parallel to B A, A C, and let them meet B T, B C in T and H. Through the point of contact B let B D be drawn, meeting the curves in D and d, and the ſtraight line E T in N; and let D C, d C be drawn, meeting E H in I and L. Then, by Cor. 4. Prop. V. H E is to E T as L E to E N, and I E is to E N in the ſame proportion, and therefore (9. v.) I E, L E are equal: which is abſurd. The two points D, d therefore coincide, and the two ſections meet in the five points A, B, D, C, E; which by this Prop. is impoſſible.

Fig. 139. *Cor.* 2. If two conic ſections A F D, A f D touch one another in the points A, D, they will not meet one another in any other point. For, if it be poſſible, let them meet one another in the point I, and let the ſtraight line I F be drawn, meeting the tangents A B, D G in B, G, and the curve of the ſection A F D in F.

As

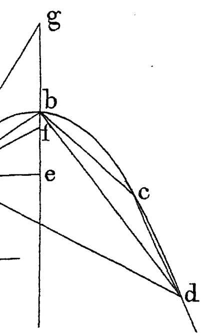


Fig. 117.

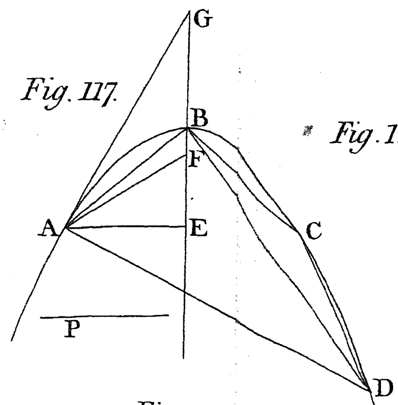


Fig. 123.

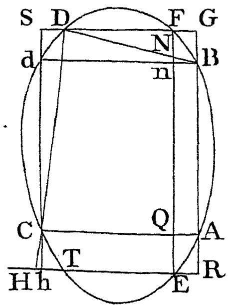
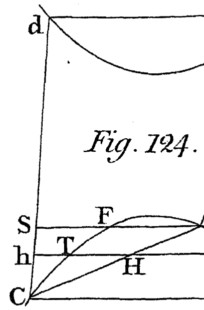


Fig. 124.



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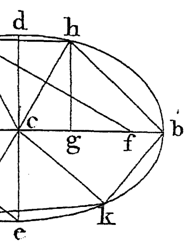


Fig. 119.

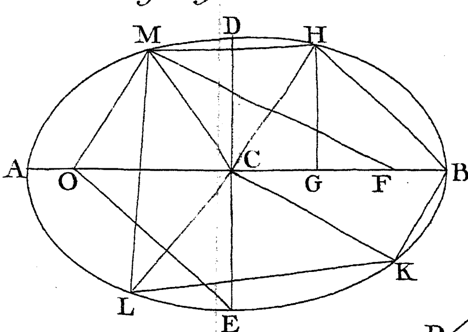


Fig. 125.

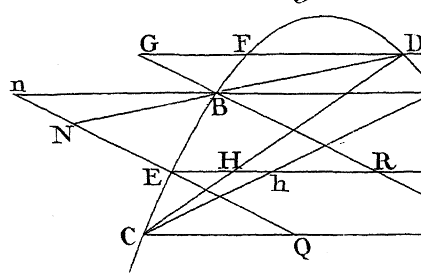


Fig. 121.

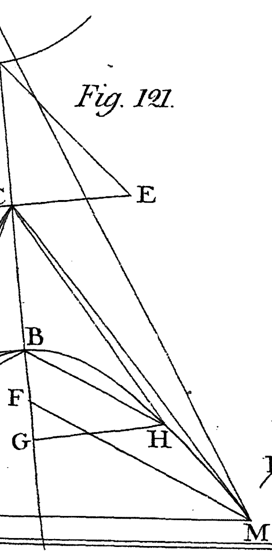
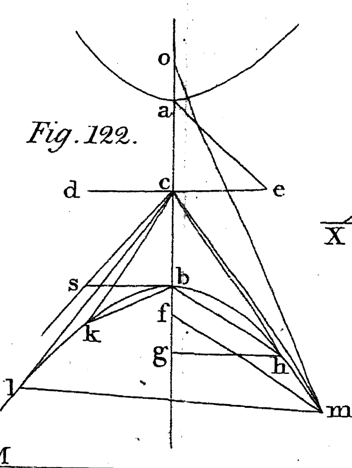


Fig. 122.



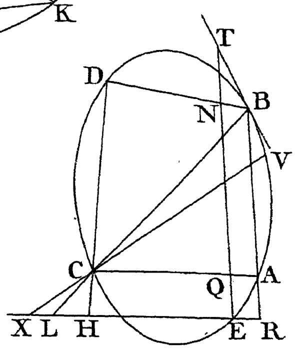
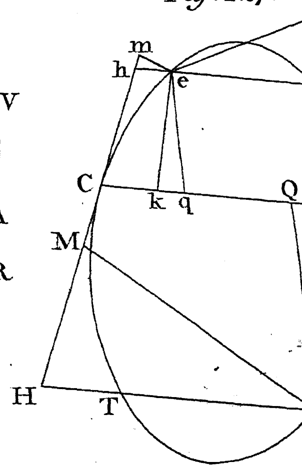
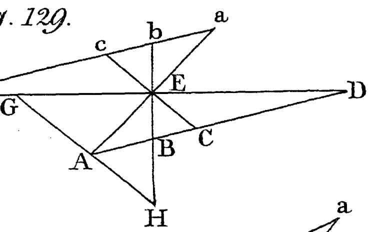


Fig. 126.

Fig. 127.





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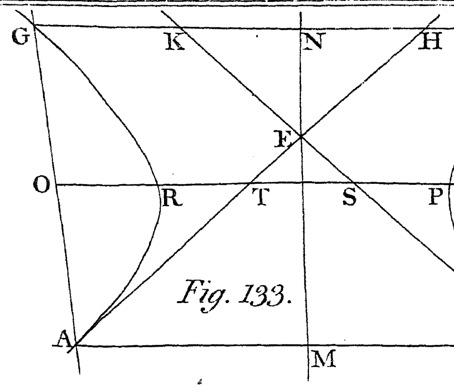

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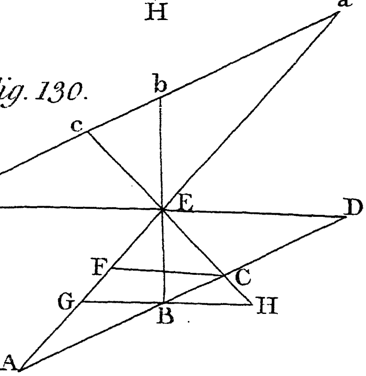


ig. 130.

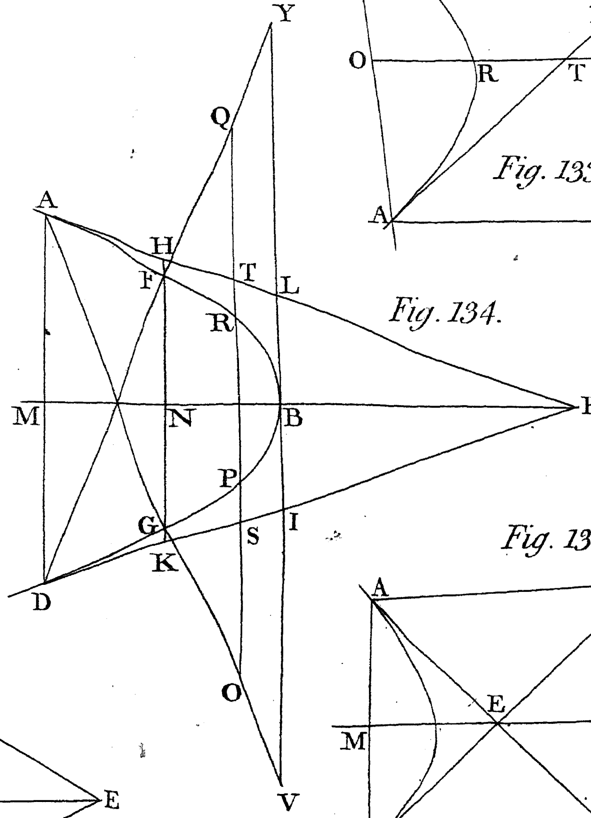
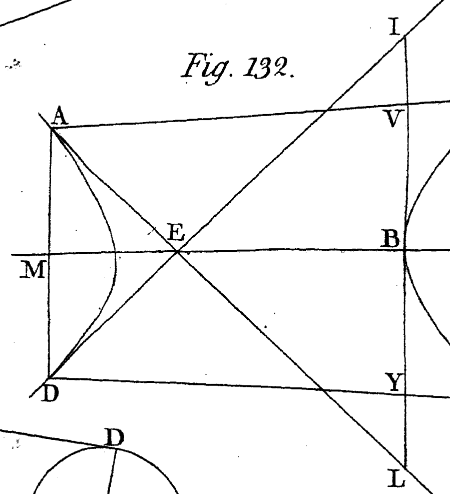


Fig. 134.



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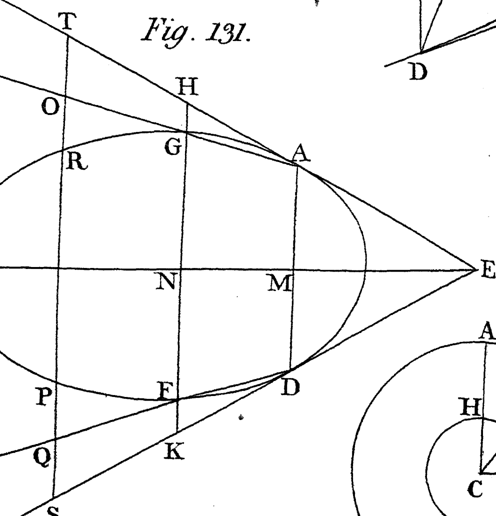

Fig. 131.

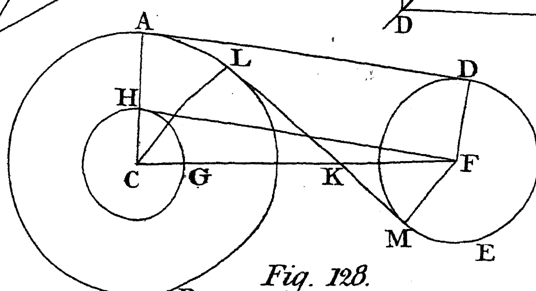

Fig. 128.

As the ſections touch one another in D, by the preceding Cor. they do not meet one another in three other points. Let the ſtraight line IG therefore meet the curve of the ſection A f D in f. Then, by Cor. 2. Prop. XVII. Book I. $BE^2 : GE^2 :: IB \times BF : FG \times GI$; and $BE^2 : GE^2 :: IB \times Bf : fG \times GI$. Conſequently (11. v.) $IB \times BF : FG \times GI :: IB \times Bf : fG \times GI$, and by alternation $IB \times BF : IB \times Bf :: FG \times GI : fG \times GI$; and therefore (1. vi.) $BF : Bf :: FG : fG$. Hence (12. v.) $BF : Bf :: BG : BG$, and therefore BF, Bf are equal: which is abſurd. Conſequently the Cor. is evident.

PROP. VIII.

If a ſtraight line touch a conic ſection, and a ſtraight line perpendicular to it be drawn through the point of contact, and meeting the axis or axes of the ſection, the ſegment of the perpendicular between the point of contact and the axis of a parabola, or between the point of contact and the tranſverſe axis of the ſection, will be the leaſt of all ſtraight lines which can be drawn from the ſame point in the axis, and on the ſame ſide of it, to the curve; but the ſegment of the perpendicular between the point of contact and the conjugate axis of the ellipſe will be the greateſt of all ſtraight lines which can be drawn from the ſame point in the axis, and on the ſame ſide of it, to the curve.

Let the ſtraight line PR touch a conic ſection in the point P, and let the ſtraight line PK, perpendicular to the tangent, meet AB the axis of a parabola, or the tranſverſe axis of the ſection, in the point K, and let it meet DE the conjugate axis of the ellipſe in M; the ſegment PK is the leaſt of all the ſtraight lines which can be drawn from K, on the ſame ſide of AB, to the curve; Fig. 140. 141. 142.

 curve;

BOOK IV. curve; and in the ellipse the segment M P is the greatest of all the straight lines which can be drawn from M, on the same side of D E, to the curve.

Let the tangent P R meet the axis A B in R; and draw P G a double ordinate to A B, and let it meet A B in F, and the curve again in G, and draw R G. Then, as the angles at F are right angles, and as P F is equal to F G, we have (4. i.) P R equal to G R, G K equal to P K, and K G R a right angle, being equal (8. i.) to the angle K P R. Let the straight line B H, touching the section in the vertex B, meet the tangent P R in H, and draw K H. Then in the parabola the square of
Fig. 140. B H is to the square of P H, as the parameter of the axis A B to the parameter of the diameter passing through P, by Prop. IV. Book III. and therefore, by Cor. 1. Prop. XI. Book III. the square of B H is less than the square of P H. Consequently, as K H is common to the two right angled triangles K B H, K P H, the square of K B (47. i.) must be greater than the square of K P, and K P must be less than K B. Again,
Fig. 141. 142. in the ellipse or hyperbola, C being the center, by Prop. V. Book II. the square of B H is to the square of P H as the square of C D to the square of the semidiameter parallel to P H; and therefore, by Prop. XI. Book II. the square of B H is less than the square of P H. Consequently, for the same reasons as in the parabola,
Fig. 140. 141. 142. P K is less than K B. If therefore with K as a center, and K P as a distance, in each section, a circle be described, its circumference will pass through G, and cut A B within the section*; and as K P R, K G R are right angles, P R and G R (16. iii.) are tangents to

* The circle is intentionally omitted in the figures. The description of it would have made them more complex, and not rendered the Proposition or either of the Corollaries more perspicuous.

the

gents to the section. Consequently, by Cor. 2. Prop. VII. the circumference of the circle cannot meet the curve of the section in any other point besides P and G; and therefore K P is the least of all straight lines which can be drawn from K, on the same side of A B, to the curve.

In the ellipse draw P N a double ordinate to the conjugate axis D E, and let it meet D E in L, and the curve again in N. Let the tangent P R meet D E in T, and draw N T, N M. Then it may be proved, as above, that M N is equal to M P, N T equal to P T, and that the angle M N T is a right angle, being equal to the angle M P T. In the ellipse let the straight line D V, touching the section in the vertex D, meet the tangent P R in V, and draw M V. Then, by Cor. 3. Prop. III. Book II. D V is parallel to the axis A B, and by Prop. V. Book II. the square of D V is to the square of P V, as the square of C B to the square of the semidiameter parallel to P V; and therefore, by Prop. XI. Book II. the square of D V is greater than the square of P V. Consequently, as the angles at D and P are right angles, and M V common to the two triangles M D V, M P V, the square of M P (47. i.) must be greater than the square of M D. If therefore with M as a center, and M P as a distance, a circle be described, its circumference will pass through N, it will cut D E without the ellipse, and, for the same reasons as above, the straight lines T P, T N will touch it and the section in the points P, N. Consequently, by Cor. 2. Prop. VII. the circumference of the circle cannot meet the curve of the section in any other point besides P and N; and therefore M P is the greatest of all straight lines which can be drawn from M, on the same side of D E, to the curve of the ellipse. Fig. 141.

Cor. 1. If a straight line as P G be a double ordinate

BOOK IV.

Fig. 140. 141. 142. to A B the axis of a parabola, or the tranſverſe axis of a conic ſection, a circle touching the ſection in P, and paſſing through G, will alſo touch the ſection in G; and the other parts of its circumference will fall wholly within the ſection. For P G will be in the circle, and, being biſected by A B at right angles, the center of the circle (Cor. 1. iii.) will be in A B. Let K be the center, and draw K P, and let P R be common to the circle and ſection, according to the third Definition. Then (18. iii.) K P R is a right angle; and K G, G R being drawn, it may be proved, as above, that the angles K G R, K P R are equal. Conſequently G R will touch the circle, (16. iii.) and it is evident, from Cor. 2. Prop. VI. Book III. that it alſo touches the ſection. Hence the Cor. is manifeſt.

Fig. 141. 142. *Cor.* 2. If a ſtraight line as P N be a double ordinate to D E the conjugate axis of an ellipſe, or of oppoſite hyperbolas, a circle touching the ellipſe or hyperbola B P in P and paſſing through N will alſo touch the ellipſe or the oppoſite hyperbola in N. For let the axis D E meet the common tangent P R in T, and the ordinate P N in L. Then, as P N will be in the circle, the center of the circle (Cor. 1. iii.) will be in D E. Let M be the center, and draw M P, M N, N T. Then (4. i.) M P is equal to M N, and T P equal to T N; and therefore (8. i.) the angle M N T is equal to the angle M P T, which is a right one. Hence (16. iii.) the circle touches the ellipſe or the oppoſite hyperbola in N, and N T is the common tangent to the circle and ſection.

In this caſe it is evident, that the circle deſcribed with the center M, and the diſtance M P, falls without the ellipſe, and without each of the oppoſite hyperbolas. For, by this Prop. M P in the ellipſe is the greateſt ſtraight line which can be drawn from M to the curve; and in the hyperbolas, as T P, T N are the common tangents, it is evident that M P, M N are the leaſt

ſtraight

If from the vertex of the axis of a parabola, or from a vertex of the transverse axis of an ellipse or hyperbola, a segment be taken in the axis equal to its parameter, a circle described about this segment as a diameter will fall wholly within the section; but if from a vertex of the conjugate axis of an ellipse a segment be taken in the axis equal to its parameter, a circle described about this segment as a diameter will fall wholly without the section.

Let A B be an axis of a conic section, and in the hyperbola the transverse axis, and from the vertex A let the segment A C be taken in the axis equal to its parameter; the circle A E C described about A C as a diameter will fall wholly within the section, unless A B be the conjugate axis of the ellipse, and if A B be the conjugate axis of the ellipse, the circle will fall wholly without the section. Fig. 143. 144. 145. 146.

For through A draw the straight line A D equal to A C, and at right angles to A B, and draw C D. Through any point F in A C draw F G an ordinate to A B, and let it meet the circumference of the circle in E, the curve of the section in G, and the straight line C D in K. In the ellipse and hyperbola draw from the vertex B the straight line B D, and let it meet F G in H; but in the parabola draw D H parallel to the axis A B, and let it meet F G in H. Then, by Prop. II. Book III. A D, F K are parallel, and therefore (4. vi.) A D : A C : : F K : F C; and as A D is equal to A C, F K is equal to F C. Consequently A F × F K is equal to A F × F C, and therefore (35. iii.) A F × F K is equal to the square of

 E F.

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E F. But, by Cor. 1. Prop. VI. Book II. and Prop. III. Book III. the ſquare of F G is equal to A F × F H. In the parabola and hyperbola therefore, and when A B is the tranſverſe axis of the ellipſe, the ſquare of F G is greater than the ſquare of F E, and conſequently the point G is without the circle. But if A B be the conjugate axis of the ellipſe, as in Fig. 146. the ſquare of F E will be greater than the ſquare of F G, and therefore the point G will be within the circle.

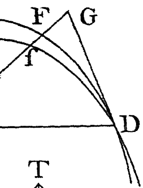
PROP. X.

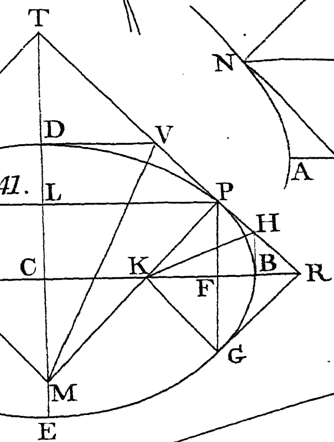
If from the vertex of the axis of a parabola, or from a vertex of the tranſverſe axis of an ellipſe or hyperbola, a ſegment be taken in the axis greater than its parameter, a circle deſcribed about this ſegment as a diameter will fall without the ſection on each ſide of the vertex; but if from a vertex of the conjugate axis of an ellipſe a ſegment be taken in the axis leſs than its parameter, a circle deſcribed about this ſegment as a diameter will fall within the ellipſe on each ſide of the vertex.

Fig. 147. 148. 149.

Firſt, let A B be the axis of a parabola, or the tranſverſe axis of an ellipſe or hyperbola, and from the vertex A let the ſegment A C be taken in the axis greater than its parameter; the circle A I C deſcribed about A C as a diameter will fall without the ſection on each ſide of the vertex A.

In the ellipſe let the point C be between the vertices A, B, and in each ſection let the ſtraight line A D be drawn perpendicular to the axis A B, and equal to its parameter. In A D let the ſegment A F be taken equal to A C, and draw C F. In the ellipſe and hyperbola let the ſtraight line B D be drawn, and let it meet C F in E; but in the parabola let the ſtraight line D E be drawn parallel to the axis, and let it meet C F in E. In each





41.

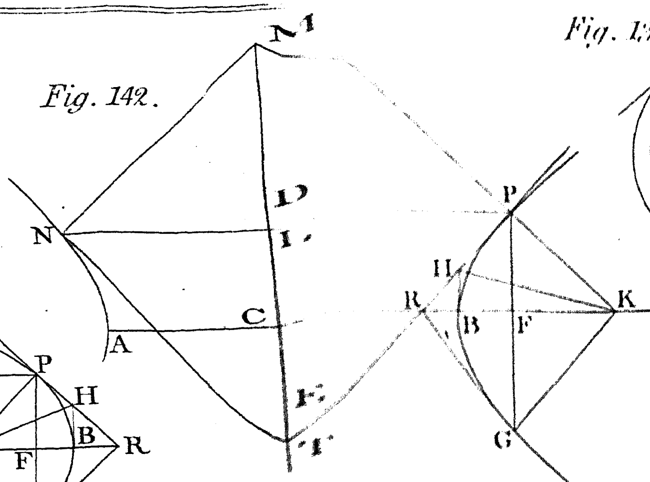


Fig. 142.

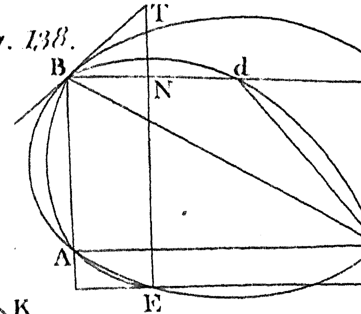


Fig. 138.

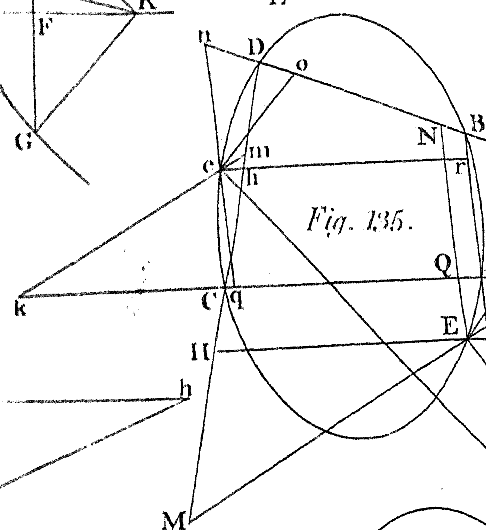


Fig. 135.

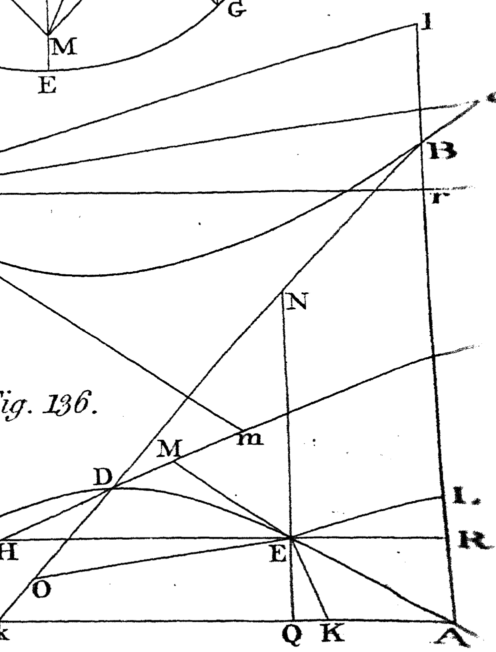


Fig. 136.

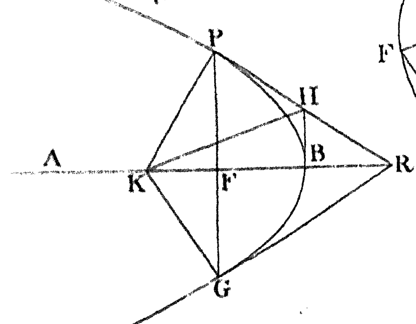


Fig. 140.

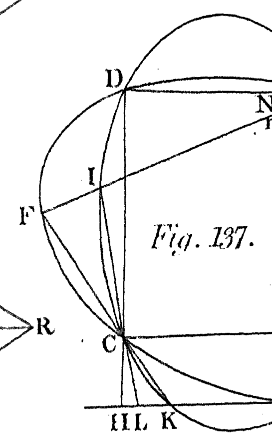


Fig. 137.

18

B

each section let any point G be taken in C F, between the points E, F; and draw G L parallel to the ordinates of A B, and let it meet D E in H, the circumference of the circle in I, the curve of the section in K, and the axis A B in L. Then, by Prop. III. Book III. and Cor. 1. Prop. VI. Book II. the square of the ordinate L K is equal to the rectangle under A L, L H; and on account of the equals A F, A C, the rectangle under A L, L G is equal to the rectangle under A L, L C, and therefore (35. iii.) equal to the square of I L. But the rectangle under A L, L G is greater than the rectangle under A L, L H, and therefore the square of I L is greater than the square of L K. Consequently the point I is without the section; and if the straight line G L meet the circumference of the circle again in M, the point M, for the same reasons as above, will be without the section. The circle A I C therefore falls without the section on each side of the vertex A.

Secondly, let A B be the conjugate axis of an ellipse, and from the vertex A let the segment A C be taken in the axis less than its parameter; the circle A I C described about A C as a diameter will fall within the ellipse on each side of the vertex A. Fig. 150.

For let A C be greater than the axis A B. Let the straight line A D be drawn perpendicular to A B, and equal to its parameter. In A D let the segment A F be taken equal to A C, and draw B D, C F, and let them meet one another in the point E. In C F take any point G between the points F, E, and draw G L parallel to the ordinates of A B, and let it meet B D in H, the curve of the ellipse in K, the axis in L, and the circumference of the circle in I, M. Then, as above, it may be demonstrated, that the points I, M are within the ellipse; and therefore that the circle A I C falls within the ellipse on each side of the vertex A.

Cor.

BOOK IV.

Cor. 1. If from the vertex of the axis of a parabola, or from a vertex of the tranſverſe axis of an hyperbola, or from a vertex of either axis in an ellipſe, a ſegment be taken in the axis equal to its parameter, a circle deſcribed about this ſegment as a diameter will have the ſame curvature with the ſection in the vertex. For, by the preceding Prop. in the parabola, and when A B is a tranſverſe axis, the circle A E C falls wholly within the ſection, the diameter A C being equal to the parameter of the axis A B; and it is evident, that if a ſegment be taken from A in A B leſs than A C, a circle deſcribed about it as a diameter will fall within the circle A E C. Again, by this Prop. if from A a ſegment be taken in A B greater than the parameter of A B, or greater than A C, in Fig. 147. it will fall without the ſection. In Fig. 143. therefore the circle A E C has the ſame curvature with the ſection in A. Alſo when A B is the conjugate axis of the ellipſe, by the preceding Prop. the circle A E C falls wholly without the ellipſe, A C being equal to the parameter of A B; and it is evident, that if a ſegment be taken from A in A B greater than A C, the circle deſcribed about it as a diameter will fall without the circle A E C. Again, by this Prop. if from A a ſegment be taken in A B leſs than the parameter of A B, or leſs than A C, in Fig. 150. a circle deſcribed about it as a diameter will fall within the ſection. In this caſe therefore the circle A E C in Fig. 146. has the ſame curvature with the ſection in A, according to the fourth Definition.

Cor. 2. If a circle touch a conic ſection in the vertex of an axis, and have the ſame curvature with the ſection in the vertex, it will cut off from the axis a ſegment equal to its parameter. This is evident from the preceding Corollary.

PROP.

PROP. XI.

If from a point in the curve of a conic section a double ordinate be drawn to the axis of a parabola, or to the transverse axis of the section, and if through the point in which it meets the curve again a diameter be drawn, and from the first mentioned point a double ordinate be drawn to this diameter; a circle touching the section in the first mentioned point, and passing through the other extremity of the last mentioned double ordinate, will not meet the section in any other point besides these two, and it will have the same curvature with the section in the point of contact.

From the point B in the curve of the conic section B A L let the straight line B A be drawn a double ordinate to D H the axis of the parabola B A L, or to D H the transverse axis of the section, and from A, the point in which it meets the curve again, draw the diameter A E, and from B draw the double ordinate B F to A E; the circle B K F, touching the section in B and passing through F, the point in which the double ordinate B F meets the curve again, will not meet the section in any other point besides B, F, and it will have the same curvature with the section in the point B. Fig. 153. 154. 155. 156.

For if the circle K F B pass through the point A, it will touch the section also in A, by Cor. 1. Prop. VIII. and the other parts of the circumference will fall wholly within the section; and if the section be an ellipse, and B E be drawn, it will be an ordinate to the conjugate axis; and therefore if the circle K F B pass through E, it will also touch the section in E, by Cor. 2. Prop. VIII. and the other parts of the circumference will fall wholly without the ellipse. If therefore the circle K F B pass through A in any section, or through

BOOK IV. through E in the ellipſe, it will touch the ſection in the points B, A or B, E, and, by Cor. 2. Prop. VII. it will not paſs through F, contrary to hypotheſis. Hence it is evident that the circle K B F in any ſection is greater than a circle which touches the ſection in B, A, but in the ellipſe it muſt be leſs than a circle which touches the ellipſe in B, E; and therefore, in any ſection, the circumference of the circle K B F meets the ſtraight line B A without, but the ſtraight line B E within the ſection. Let the circumference meet B A in K, and A E in Q. If it be poſſible, let the circumference of the circle meet the ſection in L. Draw the common tangent B D, meeting the axis D H in D, and draw A D. Then, by Cor. 2. Prop. VI. Book III. A D will touch the ſection, and therefore, by Prop. II. Book III. A D, B F are parallel, and on account of the axis D H, the tangents (4. i.) A D, B D are equal. Let L M be drawn parallel to A D or F B, meeting the curve again in I, and the tangent B D in M. Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$; and therefore BM^2 is equal to $LM \times MI$; and as, by hypotheſis, the point L is in the circumference of the circle, the point I (36. iii.) is alſo in the circumference of the circle. The circle K F B therefore, touching the ſection in B, meets the ſection in F, L, I, which, by Cor. 1. Prop. VII. is impoſſible. The circle K F B therefore does not meet the ſection in any point beſides B, F; and as the point K is without, and the point Q within the ſection, the arch B K F will be without, and the arch F Q B will be within the ſection.

The circle K F B will alſo have the ſame curvature with the ſection in the point B. For let any other circle as B R L, touching the ſection in B, be deſcribed; and, firſt, let B R L be leſs than K F B. Then, as the ſtraight

in the same point B, the centers of these circles (13.
iii.) will be in the ſame ſtraight line, and therefore the lesser circle B R L will fall wholly within the circle K F B. The circle B R L therefore cannot paſs between the arch B Q F and the curve of the ſection; and if it paſs through A, or be leſs than a circle paſſing through A, it will fall wholly within the ſection, by Cor. 1. Prop. VIII. and therefore it cannot paſs between the arch F K B and the curve of the ſection. But let the circle B R L be greater than the circle paſſing through A, and meet the ſtraight line B A in V, and the curve of the ſection in L*. Let the ſtraight line L M be drawn parallel to A D, meeting the curve of the ſection again in I, and the tangent B D in M. Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$, and, on account of the equals B D, A D, the ſquare of B M is equal to $LM \times MI$. The point I therefore (36. iii.) is in the circumference of the circle, and conſequently, by Cor. 1. Prop. VII. it meets the ſection only in the points B, L, I. Again, it is evident that the circle B R L meets B F within, and B A without the ſection, and therefore that the curve of the ſection between the points F, L are without the circle. In the tangent B M therefore take any point T between B and M, and let the ſtraight line T P be drawn parallel to A D, meeting the ſection in P, s, and the circle B R L in R, p. Then, as above, it may be demonſtrated that BT^2 is equal to $PT \times Ts$. But the ſquare of B T (36. iii.) is equal to the rectangle $RT \times Tp$, and therefore the rectangles $PT \times Ts$, $RT \times Tp$ are equal. Conſequently $PT : TR :: Tp : Ts$, and as P T is greater than T R, Tp is greater than Ts. The arch B p I therefore falls within the ſection. Fig. 153. 154. 156.

* Such lines as could eaſily be ſupplied by the mind of the reader are omitted in Fig. 155. and 156.

Se-

BOOK IV.

Fig. 155. Secondly, let the circle B R L be greater than the circle K F B; and it may then be demonstrated, as above, that the circle B R L falls without K F B, and therefore that it cannot pass between the arch F K B and the section. Again, if the circle B R L pass through E in the ellipse, or be greater than the circle passing through E, it will fall wholly without the ellipse, and therefore it will not pass between the ellipse and the arch F Q B. But in any section let the circle B R L meet A E within the section, and the curve of the section in L. Draw the straight line L M parallel to A D, and let it meet the section in I, and the tangent B D in M. In B D take any point T between B and M. Draw T R, and let it meet the section in P and s, and the circle B R L in R and p. Then, as above, it may be demonstrated that the arch P I is without the section. The circle K F B therefore has the same curvature with the section in the point of contact B, according to Def. IV.

Cor. From the above, and Cor. 1. Prop. X. it is evident that only one circle, touching a conic section in a given point, can have the same curvature with the section in that point.

PROP. XII.

If a circle touch a conic section, and have the same curvature with the section in the point of contact, it will cut off from the diameter of the section passing through the point of contact a segment equal to its parameter.

If the point of contact be the vertex of an axis, the Proposition has been demonstrated, as stated in Cor. 2. Prop. X. but if the circle K B F touch the section in the point B, which is not a vertex of an axis, let every thing remain as in the preceding Prop. and let K B F be the circle having the same curvature with the section in the

Fig. 160. 161. 162.

the

the point B. Let the circumference of the circle K B F therefore meet B C, drawn through B the point of contact and C the center, in the point N, if the ſection be an ellipſe or hyperbola; or let it meet the diameter B N in the point N, if the ſection be a parabola; in either caſe the ſegment B N is equal to the parameter of the diameter B C N.

Firſt, let the ſection be an ellipſe or hyperbola, and draw the diameter C V parallel to the tangent D A, and the diameter T M parallel to the tangent B D, and meeting B F in G. Let the tangent D A meet the diameter T M in M, and the diameter B C in L. Let the diameter A E meet its ordinate B F in I. Then, on account of the axis D C and its ordinate A B, the ſemidiameters C A, C B are equal; and by ſimilar triangles L A : A C :: B I : C I; and A C : A M :: C I : I G. Conſequently, Fig. 160. 161.

L A : A C : A M

B I : C I : I G,

and (22. v.) L A : A M :: B I : I G; and therefore (22. vi.) L A × A M : B I × I G :: $LA^2 : BI^2$, or, by the above, as AC^2 to CI^2. But, by Cor. 2. Prop. IV. Book II. the diameters T M, B L are conjugate, and therefore L A × A M is equal to CV^2, by Cor. 2. Prop. IX. Book II. Conſequently CV^2 : B I × I G :: AC^2 : CI^2; and, by alternation, $CV^2 : AC^2$ :: B I × I G : CI^2; and, by Prop. V. Book II. $CV^2 : AC^2 :: BI^2$: A I × I E. Conſequently, by the tenth Lemma, (and 12. v. and 3. ii.) $CV^2 : AC^2$ :: G B × B I : AC^2, and therefore (14. v.) CV^2 is equal to G B × B I. But, by Prop. V. Book II. $AD^2 : BD^2 :: CV^2 : CT^2$; and therefore, on account of the equals A D, B D, the ſquare of C V is equal to the ſquare of C T. The ſquare of C T is therefore equal to G B × B I; and N B × B C is equal to F B × B G, by the ſeventh Lemma. If therefore N B be biſected in P, the rectangle under P B, B C will

BOOK IV.

will be equal to $GB \times BI$, or to CT^2. But CT^2 is equal to the rectangle under CB and half the parameter of CB. Consequently the segment BN is equal to the parameter of the diameter CB.

Fig. 162. Secondly, let the section be a parabola, and let the diameter AE meet its ordinate BF in E. Draw to the diameter BN the ordinate AC, meeting the diameter BN in C, and BF in H. Then, on account of the equals AD, BD, the parameter of the diameters AE, BC will be equal, by Prop. IV. Book III. and by Cor. Prop. V. Book III. AE, BC are equal, and as AE, BC are parallel, the triangles AEH, BCH are equiangular. Consequently (4. vi.) AE : EH :: BC : BH, and therefore (14. v.) EH, HB are equal. Consequently FB : BE :: BE : BH, and therefore $FB \times BH$ is equal to BE^2. But, by the seventh Lemma, $FB \times BH$ is equal to $NB \times BC$; and as BE, AC are equal, BE^2 is equal to AC^2. The rectangle NBC therefore is equal to the square of AC. Moreover the square of AC is equal to the rectangle under BC, and the parameter of the diameter BN; and therefore the rectangle NBC is equal to the rectangle under the abscifs BC, and the parameter of the diameter BN. The segment BN is therefore equal to the parameter of the diameter drawn through B the point of contact.

Cor. 1. If from the vertex of a diameter of a parabola, or from a vertex of a transverse diameter of an hyperbola, or from a vertex of any diameter of an ellipse, a segment be taken in the diameter equal to its parameter, a circle touching the section in the vertex, and passing through the other extremity of the segment, will have the same curvature with the section in the vertex. This is evident from Cor. 1. Prop. X. and the above.

Cor. 2. If through O, the focus of the parabola, the straight

meter of the diameter B C.

For through O draw O X parallel to B D, and let it meet B C in X. Then, by Cor. Prop. XII. Book III. B X is equal to B O; and by the ſeventh Lemma R B × BO is equal to N B × B X. Conſequently B R is equal to B N.

Cor. 3. The reſt remaining as above in the ellipſe and hyperbola, if the ſtraight line B R drawn through the focus O meet the circle again in R, and X S be the tranſverſe axis, then B R will be to the diameter C T as C T is to X S. For let the diameter C T meet B R in Y, and then, by the ſeventh Lemma, the rectangle R B Y is equal to the rectangle F B G. But by the above the rectangle F B G is equal to twice the ſquare of C T, and therefore the rectangle R B Y is equal to twice the ſquare of C T. Conſequently R B : the whole diameter C T : : the ſemidiameter C T : B Y. But, by Cor. Prop. XVI. Book II. B Y is equal to C X, and therefore (15. v.) B R : the whole diameter C T : : the whole diameter C T : X S. Fig. 160. 161.

PROP. XIII.

If a circle touching an ellipſe or hyperbola have the ſame curvature with the ſection in the point of contact, its ſemidiameter will be to the ſemidiameter of the ſection conjugate to that paſſing through the point of contact, as the ſquare of the ſame ſemidiameter of the ſection to the rectangle under the ſemiaxes.

For if the circle touch the ſection in the vertex of an axis, then every thing remaining as in Prop. IX. by the Definition of a parameter, and inverſion, A C is to the axis parallel to the common tangent at A, as the ſame axis to the axis A B. Conſequently (15. v. and Fig. 144. 145. 146.

BOOK IV. 1. iv.) as the ſemidiameter of the circle A E C to the ſemiaxis parallel to the common tangent at A, ſo is the ſquare of the ſame ſemiaxis to the rectangle under the ſemiaxes.

Fig. 160. 161. But if the circle of curvature do not touch the ſection in the vertex of an axis, let every thing remain as in Prop. XII. and let U B be the ſemidiameter of the circle; and then U B will be to C T as CT^2 to the rectangle under the ſemiaxes.

For let C X be the tranſverſe ſemiaxis, and C H the conjugate ſemiaxis. From the center C draw the perpendicular C Q to the tangent B D. Let U B meet the circumference again in W, and draw W N. Then (31. iii.) the angle W N B is equal to the angle C Q B, and the angle N B W (29. i.) equal to the angle B C Q. Conſequently (4. vi.) C B : C Q : : W B : B N, or (15. v.) as U B to P B; and therefore C Q × U B is equal to C B × P B. But, by Prop. XII. and the Definition of a parameter, C B × P B is equal to CT^2, and therefore U B : C T : : C T : C Q. Conſequently (1. vi.) U B : C T : : CT^2 : C T × C Q. But, by Cor. 1. Prop. XIX. Book II. C T × C Q is equal to C X × C H; and therefore U B : C T : : CT^2 : C X × C H.

Cor. By the above the ſquare of C T is equal to the rectangle under U B, C Q.

PROP. XIV.

If from the center of a circle, touching a conic ſection, and having the ſame curvature with the ſection in the point of contact, a perpendicular be let fall upon a ſtraight line drawn from the point of contact through the neareſt focus, a ſtraight line drawn from the point of concourſe to the point in which the diameter of the circle, paſſing through the point of contact, cuts the focal axis will be at right angles to this diameter of the circle.

From

From H the center of the circle L P M, touching the conic ſection A P in the point P and having the ſame curvature with the ſection in P, let the perpendicular H K be drawn to the ſtraight line P K paſſing through F the neareſt focus; the ſtraight line K I, drawn from K the point of concourſe to I the point in which the diameter P H of the circle cuts the focal axis A I, is at right angles to H P. BOOK IV. Fig. 157. 158. 159.

For firſt let the ſection be an ellipſe or hyperbola of which C is the center, A C the tranſverſe ſemiaxis, and D C the conjugate ſemiaxis. Let N P be the common tangent, and C T the ſemidiameter of the ſection parallel to N P; and draw C G, F N perpendicular to N P. Then, by Cor. Prop. XIII. the ſquare of C T is equal to the rectangle under H P, C G; and, by Prop. XVII. Book II. the ſquare of C D is equal to the rectangle under I P, C G. Conſequently $CT^2 : CD^2 :: HP \times CG : IP \times CG$; and therefore (1. vi.) $CT^2 : CD^2 :: HP : IP$. But, by Prop. XIX. Book II. (and 22. vi.) $CT^2 : CD^2 :: FP^2 : FN^2$; and, on account of the ſimilar triangles, $FP^2 : FN^2 :: HP^2 : PK^2$. Conſequently (11. v.) $HP^2 : PK^2 :: HP : IP$; and therefore (1. vi.) $HP^2 : PK^2 :: HP^2 : HP \times IP$, and (14. v.) PK^2 is equal to $HP \times IP$. Hence $HP : PK :: PK : IP$, and (6. vi.) P I K is a right angle. Fig. 157. 158.

Secondly, let the ſection be a parabola, and let the common tangent N P meet the axis A I in G. Then, by Cor. Prop. IX. Book III. G F is equal to F P, and therefore (6. i.) the angle F G P is equal to the angle F P G. But as each of the angles G P I, H K P is a right angle, the angles F P G, F P I together are equal to the angles K P I, P H K together. Conſequently the angle I G P is equal to the angle P H K; and therefore (4. vi.) $GI : IP :: PK : PH$, and $HP \times PI$ is equal to $GI \times PK$. But, by Cor. 2. Prop. XII. (and 3. ii.) P K is Fig. 159.

 equal

BOOK IV. equal to half the parameter of the diameter paſſing through P, and therefore, by Prop. XI. Book III. P K is equal to G I. Conſequently H P × P I is equal to P K². As above, therefore, H P : P K : : P K : I P, and (6. vi.) P I K is a right angle.

Fig. 157. 158. 159. *Cor.* If a ſtraight line N G touch a conic ſection in P, and P H at right angles to it meet the focal axis A B in I, and if I K perpendicular to P H meet in the point K the ſtraight line P L drawn through the neareſt focus F, and, laſtly, if K H at right angles to P L meet P H in H, the point H will be the center of the circle which touches the ſection, and has the ſame curvature with it in P.

SCHOLIUM.

Fig. 160. If a body B revolve in a ſpace void of reſiſtance about the center C in the curve B A F, and if the circle B K F touch the curve in the point B, and have the ſame curvature with it in that point; and if the ſtraight line B C N meet the circle again in N, and C Q be perpendicular to the common tangent B Q, Sir Iſaac Newton has demonſtrated, in Cor. 3. to Prop. VI. Lib. I. of the Principia, that the centripetal force is reciprocally as $CQ^2 \times NB$, or directly as $\frac{1}{CQ^2 \times NB}$.

By means of this expreſſion, and the properties of oſculating circles demonſtrated in the preceding Propoſitions, the centripal forces of bodies moving in conic ſections may be eaſily aſcertained, as in the following examples.

Fig. 160. 1. Let the body revolve in the ellipſe X H S, and let the law of centripetal force tending to C the center be required.

Every thing remaining as in Prop. XIII. the centripetal force, according to the Newtonian expreſſion, is reciprocally as $CQ^2 \times BN$. But, by Cor. 1. Prop. XIX.

XIX. Book II. $CT \times CQ = XC \times CH$, and therefore $CQ^2 = \frac{XC^2 \times CH^2}{CT^2}$; and by the Definition of a parameter, and Prop. XIII. $BN = \frac{4CT^2}{2CB} = \frac{2CT^2}{CB}$. Consequently $CQ^2 \times BN = \frac{XC^2 \times CH^2}{CT^2} \times \frac{2CT^2}{CB} = \frac{XC^2 \times 2CH^2}{CB}$; and therefore, as $XC^2 \times 2CH^2$ is a constant quantity, the centripetal force is reciprocally as $\frac{1}{CB}$, or directly as the distance CB. BOOK IV.

2. Let a body P move in an ellipse or hyperbola PA, and let the centripetal force tending to the focus F of the section be required.

The rest remaining as in Prop. XIV. let the straight line PF meet the circle again in L, and, by Cor. 3. Prop. XII. $LP : 2CT :: 2CT : 2CA$; and therefore $LP = \frac{4CT^2}{2CA} = \frac{2CT^2}{CA}$. Again, by Prop. XIX. Book II. (and 22. vi.) $FN^2 : FP^2 :: CD^2 : CT^2$, and $FN^2 = \frac{FP^2 \times CD^2}{CT^2}$. But, the Newtonian general expression being adapted to the present Figures, the centripetal force is reciprocally as $FN^2 \times LP$; and therefore this force, by the above, is reciprocally as $\frac{FP^2 \times CD^2}{CT^2} \times \frac{2CT^2}{CA} = \frac{FP^2 \times 2CD^2}{CA}$. Again, if the letter L be put for the parameter of the transverse axis, $2AC : 2CD :: 2CD : \frac{4CD^2}{2AC} = \frac{2CD^2}{AC} = L$. Consequently as L, or its value, is constant, the centripetal force is reciprocally as FP^2, or inversely as the square of the distance. Fig. 157. 158.

3. Let a body P move in the curve of a parabola PA, and let the law of centripetal force tending to the focus F be required.

 Every

BOOK IV.

Fig. 159. Every thing remaining as in Prop. XIV. let the ſtraight line P F meet the circle again in L, and let F N be drawn perpendicular to the tangent P G. Then, by Cor. 2. Prop. XII. of this, and Cor. 2. Prop. XI. Book III. $LP = 4 FP$; and, by Prop. X. Book III. $FN^2 = FP \times AF$. Again by the Newtonian general expreſſion, adapted to this Figure, the centripetal force is reciprocally as $FN^2 \times LP$. By the above therefore the centripetal force is reciprocally as $FP \times AF \times 4 FP = FP^2 \times 4 AF$. Conſeqnently, as 4 A F is conſtant, the centripetal force is reciprocally as FP^2.

PROP. XV.

If three ſtraight lines touch a conic ſection, or oppoſite hyperbolas, any one of them will be harmonically divided in its point of contact, the points in which it meets the other two, and the points in which it meets the ſtraight line joining their points of contact.

Fig. 151. 152. 165. 166. 167. Let the three ſtraight lines Q R, R P, P N touch a conic ſection Q E N, or oppoſite hyperbolas Q, N in the points Q, E, N; any one of them as R P is harmonically divided in its point of contact E, the points R, P in which it meets the other tangents, and the point A in which it meets the line Q N joining their points of contact.

Fig. 151. 152. Caſe 1. Firſt, let the tangents Q R, N P be parallel; and then, by the Cor. to Prop. XIII. Book I. R E : E P : : Q R : N P. But (4. vi.) Q R : N P : : R A : A P, and therefore (11. v.) R A : A P : : R E : E P.

Fig. 165. 166. 167. Caſe 2. Let the tangents Q R, N P meet one another in the point B; and through the point P draw P H parallel to Q R, and let it meet the curve of the ſection in G, H, and Q N in I. Then, by Prop. XIII. Book I. $RE^2 : EP^2 :: QR^2 : HP \times PG$. But, by Prop. XVII. Book I. $HP \times PG$ is equal to PI^2. Conſequently $RE^2 :$

EP^2

$EP^2 :: QR^2 : PI^2$, and (22. vi.) $RE : EP :: QR : PI$. But (4. vi.) $QR : PI :: RA : AP$, and therefore (11. v.) $RA : AP :: RE : EP$. BOOK IV.

PROP. XVI.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, a ſecant paſſing through the point of concourſe will be harmonically divided in the point of concourſe, the points in which it meets the curve or curves, and the point in which it meets the ſtraight line joining the points of contact.

Let the two ſtraight lines EA, FA, touching the ſection EDF, or the oppoſite hyperbolas E, F, in the points E, F, meet one another in A, and let the ſtraight line AB meet the curve or curves in B, D, and the ſtraight line EF in C; the ſtraight line AB is harmonically divided in the points A, D, C, B. Fig. 151. 152. 168. 169.

For through B, D draw the ſtraight lines GM, HK parallel to EF, and let them meet the tangents EA, FA in G, M, and H, K and the curve or curves in B, L, and D, I. Then, by Prop. IV. GB, LM are equal, and therefore GL is equal to BM; alſo HD, IK are equal, and therefore HI is equal to DK. By equiangular triangles GB is to HD as AB to AD, and BM, or its equal GL, is to DK, or its equal HI, in the ſame proportion. Conſequently (11. v.) $GB : HD :: LG : IH$, and by alternation $GB : LG :: HD : IH$; and therefore (22. vi.) $BG \times GL : DH \times HI :: GB^2 : HD^2$. But, on account of the parallels, $GB^2 : HD^2 :: AB^2 : AD^2$; and, by Prop. XIII. Book I. $BG \times GL : DH \times HI :: EG^2 : EH^2$. Alſo, on account of the parallels, (10. vi.) $EG^2 : EH^2 :: CB^2 : CD^2$; and therefore (11. v.) $AB^2 : AD^2 :: CB^2 : CD^2$. Conſequently (22. vi.) $AB : AD :: CB : CD$.

If AB biſect EF in C, by Cor. 1. to Prop. V. Book III.

BOOK IV.

III. it will be a diameter. If therefore in this caſe A B meet the curve of the ſection in two points, or the curve of each of the oppoſite hyperbolas in one, it muſt be a diameter of an ellipſe, or a tranſverſe diameter of an hyperbola; as a diameter of a parabola can meet the curve in one point only, and in the hyperbola a ſecond diameter does not meet either of the oppoſite curves. Conſequently, by Cor. 3. Prop. VII. Book II. A B : A D : : B C : C D.

PROP. XVII.

If two ſtraight lines touching a conic ſection, or oppoſite hyperbolas, meet one another, and a ſecant paſs through the point of concourſe, tangents paſſing through the points in which the ſecant meets the curve or curves will either be parallel, or they will meet one another in the ſtraight line joining the points of contact of the two firſt mentioned tangents.

Fig. 170. 171. 172. 173.

Let the ſtraight lines K L, K M, touching the conic ſection L G M, or the oppoſite hyperbolas L, M in the points L, M, meet one another in K, and let the ſtraight line K B meet the curve, or oppoſite curves, in the points G, B; ſtraight lines touching the curve or curves in G and B will either be parallel, or they will meet in the ſtraight line L M.

For if the ſtraight line K B biſect L M, by Cor. 1. Prop. VI. Book III. K B is a diameter, and L M is a double ordinate to it; and, by Prop. II. Book III. tangents paſſing through G, B will be parallel to L M, and therefore parallel to one another. But let K B meet L M in H, and not biſect it. Let the tangents G N, B N be drawn, and meet one another in N, and if it be poſſible let N not be in the ſtraight line L M. Draw N L, and let it meet K B in V. Let the tangents N G, N B meet the tangent K L in S and T. Then, by Prop. XV.

T K

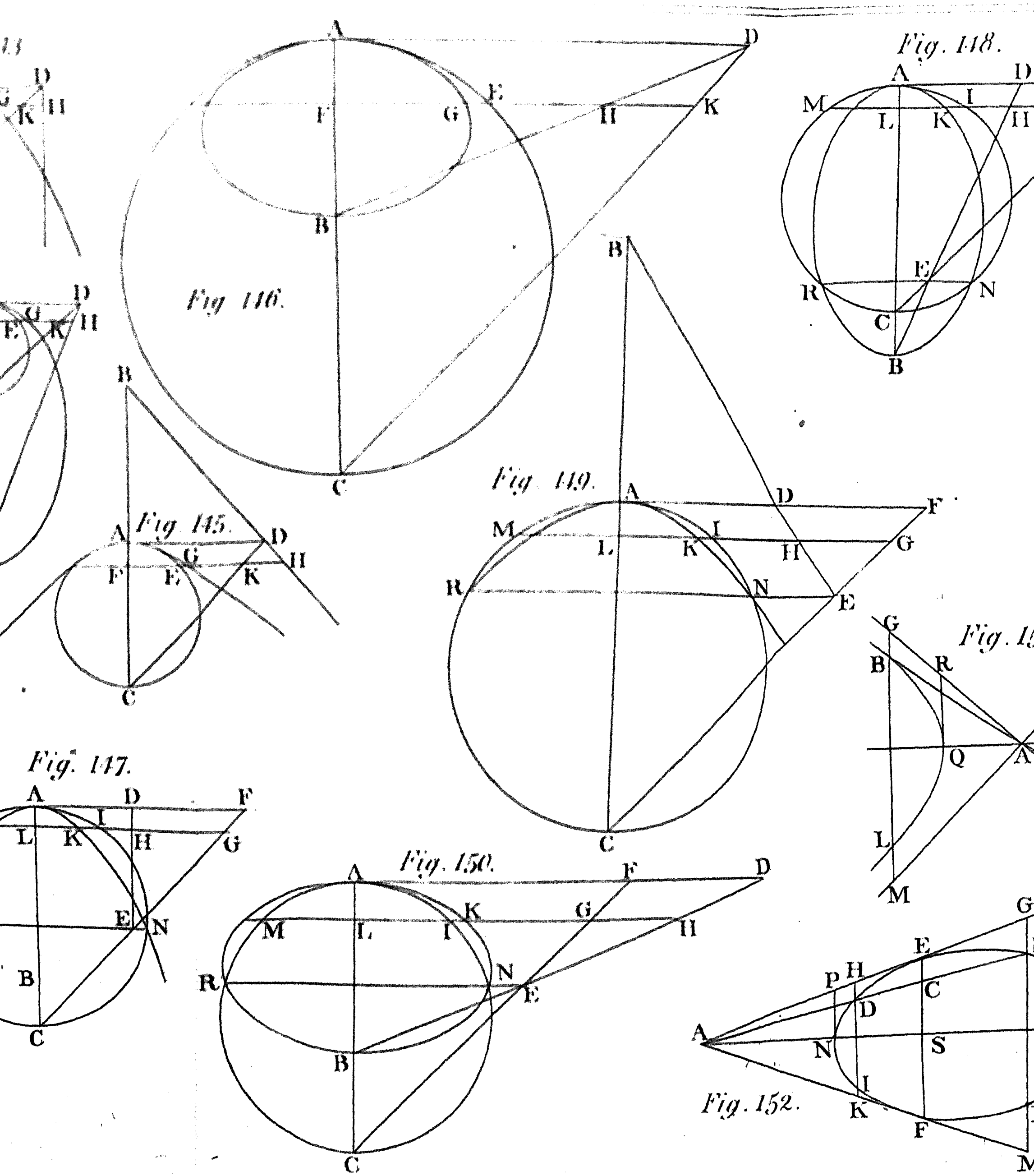


Fig. 145.
Fig. 146.
Fig. 147.
Fig. 148.
Fig. 149.
Fig. 150.
Fig. 152.

.153.

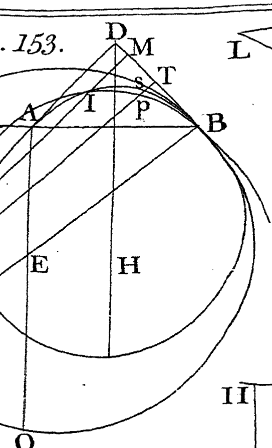


Fig.159.

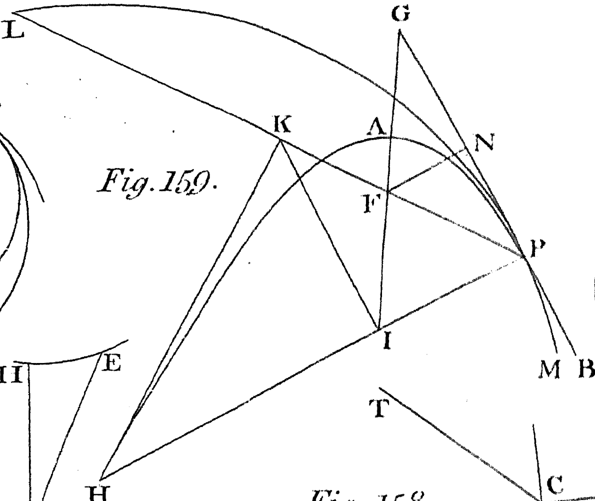


Fig.155.

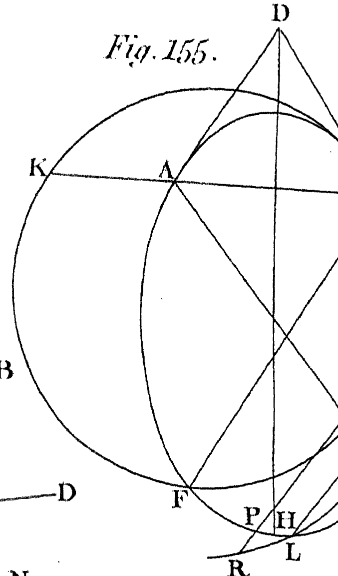


Fig.156.

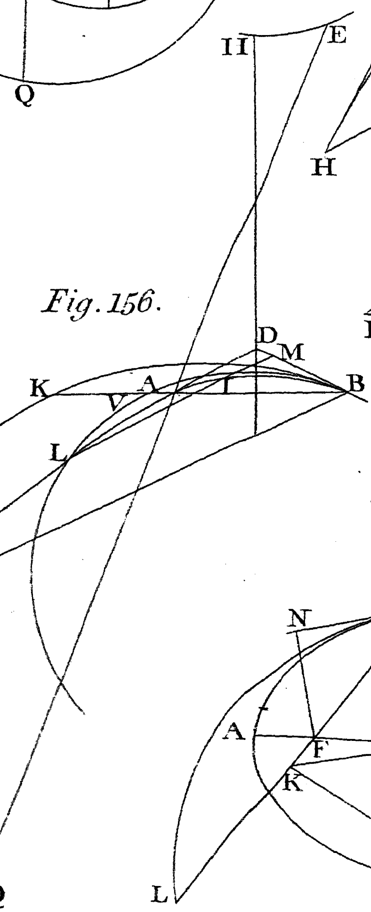


Fig.158.

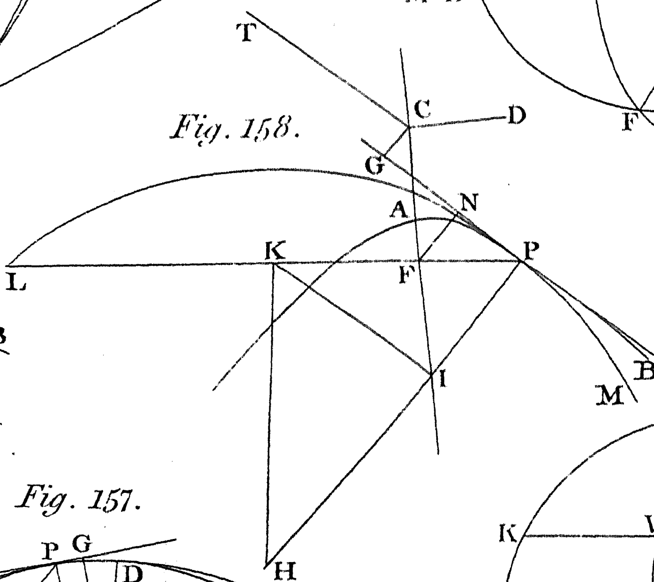


Fig. 157.

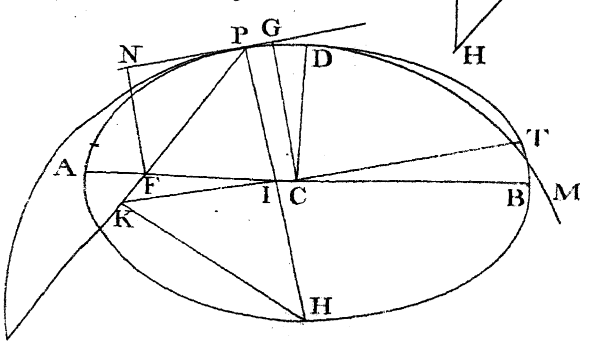
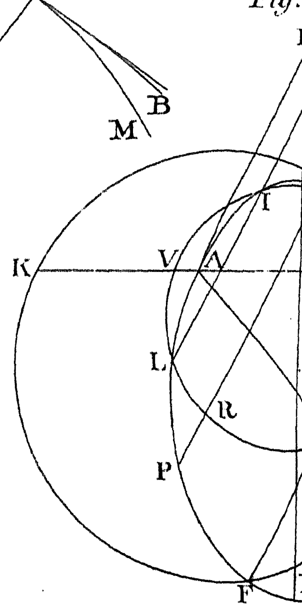


Fig.



TK : KS : : TL : LS; and therefore, on account of the harmonicals NK, NG, NV, NB, by the twelfth Lemma, BK : KG : : BV : VG. But, by Prop. XVI. BK : KG : : BH : HG; and therefore (11. v.) BV : VG : : BH : HG. Consequently (17. and 18. v.) BG : VG : : BG : HG, and (14. v.) VG is equal to HG; which is absurd. The tangents GN, BN meet therefore in the straight line LM.

Cor. 1. If a straight line as LM, not passing through the center, cut a conic section, or opposite hyperbolas, in L, M, and from points N, D, &c. in LM, tangents NB, NG, DE, DF, &c. be drawn to the curve, or opposite curves, the straight lines BG, FE, &c. each joining the points of contact of two tangents drawn from the same point in LM, will meet one another in the point K, in which the tangents passing through L, M meet one another.

Cor. 2. If the straight lines BG, FE, not passing through the center, meet the curve, or opposite curves, in B, G and F, E, and one another in K, and if they be harmonically divided in B, H, G, K and F, Q, E, K; then tangents passing through B, G, or through F, E, will meet one another in the straight line drawn through H, Q.

Cor. 3. If the tangents BN, GN meet in N, and the tangents FD, ED in D, and the straight lines BG, FE joining the points of contact meet one another in K, and the straight line ND in H and Q; the straight lines KB, KF will be harmonically divided in K, G, H, B and in K, E, Q, F.

PROP. XVIII.

If the opposite sides of a trapezium inscribed in a conic section be not parallel, and neither pass through the center, the intersection of straight lines joining the opposite angular points, the intersection of two of the opposite sides, and

BOOK IV.

and the interfections fituated between thefe fides, of tangents paffing through the points in which thefe fides meet the curve or curves, will be all four in the fame ftraight line.

Fig. 164. Let G B F E be a trapezium infcribed in a conic fection, or oppofite hyperbolas, having no two of its oppofite fides parallel, and no one of them paffing through the center; the interfection I of the ftraight lines B E, F G joining the oppofite angular points, the interfection of the oppofite fides B F, G E, and the interfections D, N, fituated between G V, B V, of the tangents E D, F D, and G N, B N, are all four in the fame ftraight line.

Let the fides B G, F E meet one another in K; and N D being drawn, let it meet K B in M, K F in L, and B F in V. Then, by Cor. 3. Prop. XVII. K B is harmonically divided in K, G, M, B, and K F in K, E, L, F. Firft, if the ftraight line G E do not pafs through V, draw V K, V E; and let V E meet K B in Q. Then, on account of the harmonicals V K, V Q, V M, V B, the ftraight line K B will be harmonically divided in K, Q, M, B; which is abfurd, by Cor. 2. to the firft Definition before Lemma XI. Confequently B F, G E meet one another in the ftraight line N D. Secondly, if N D do not pafs through the point I, draw I K, M I, and let M I meet K F in P. Then, on account of the harmonicals I K, G F, M P, B E, the ftraight line K F is harmonically divided, by Lemma XII. in K, E, P, F; which, by Cor. 2. to the firft Definition, before Lemma XI. is abfurd. Confequently the ftraight lines B E, G F meet one another in the ftraight line N D; and I, V, D, N the points of the interfections are in the fame ftraight line N D.

SCHOLIUM.

In the following Problems, when the expreffion Art. with

with a figure occurs, the article so numbered in the Scholium at the end of the third Book is referred to. BOOK IV.

The first six of the following solutions, although in several respects different, apply to the 59th, 60th, and 61st Problems in the Arithmetica Universalis, and also to the 22nd, 23rd, 24th, 25th, 26th, and 27th propositions in the first Book of the Principia.

PROP. XIX. PROB. I.

Given five points in the curve of a conic section, to describe the section.

Let E, A, B, C, D be five points given in the curve of a conic section, to describe the section. Fig. 163.

Draw A B, B D, D C, C A, B C; and through the point E draw E L parallel to A C, and E T parallel to A B. Let the straight line E L meet B C in L, and the straight line D C in H; and let E T meet B D in N. In E T, and on the same side of E H with N, take the segment E T so that H E may be to E N as L E to E T. Then B T being drawn it will touch the section, by Cor. 3. Prop. V. In the same way straight lines D F, A G may be drawn touching the section in D, A. Then if any two of the tangents, suppose D F, B T be parallel, the section will be an ellipse, according to Prop. VIII. Book I. and D B will be a diameter according to Cor. 1. Prop. II. Book II. In this case if a straight line be drawn from E parallel to D F, or B T, it will be an ordinate to B D; and the conjugate diameter to B D being found by art. 10. the section may be described.

But if no two of the tangents be parallel to one another, let B T meet D F in F, and A G in G. Draw F K bisecting B D in K, and G I bisecting A B in I. Then, by Cor. 1. Prop. VI. Book III. F K, G I will be dia-

diameters, and therefore if they are parallel, the section will be a parabola: but if they are not parallel, they will meet in the center of the section.

If the section be a parabola let F K be bisected in P, and then it is evident, by Prop. V. Book III. and art. 9. that the section may be described. But if the section be not a parabola, let F K, G I meet in O, and O will be the center. Between F O, O K let a mean proportional O P be found, and, by Prop. VII. Book II. O P will be a semidiameter of the section, to which B D is a double ordinate. Consequently the diameter conjugate to O P being found by art 10. the section may be described.

PROP. XX.

Fig. 176. 177. *If the straight lines* A C, B D *cutting a conic section in* A, C *and* B, D, *and meeting one another in* G, *meet in* T *and* F *a straight line* T F *which touches the section in* E, *the square of* E T *will be to the square of* E F *in the ratio compounded of the ratio of the rectangle under* B G, G D *to the rectangle under* A G, G C *and of the ratio of the rectangle under* A T, T C, *to the rectangle under* B F, F D.

Fig. 176. First let the section be a parabola or hyperbola, and let the point T be on that side of the figure on which the section can be extended. Let the straight line I K be drawn parallel to B D, and let it meet the curve in I, K; and let the square of the straight line V X be equal to the rectangle I T K. Then, by Prop. XIII. Book I. the square of E T is to the square of E F as the square of V X to the rectangle B F D. But the square of V X is to the rectangle B F D in the ratio compounded of the square of V X to the rectangle A T C and of the ratio of A T C to the rectangle B F D; that

Fig. 160.

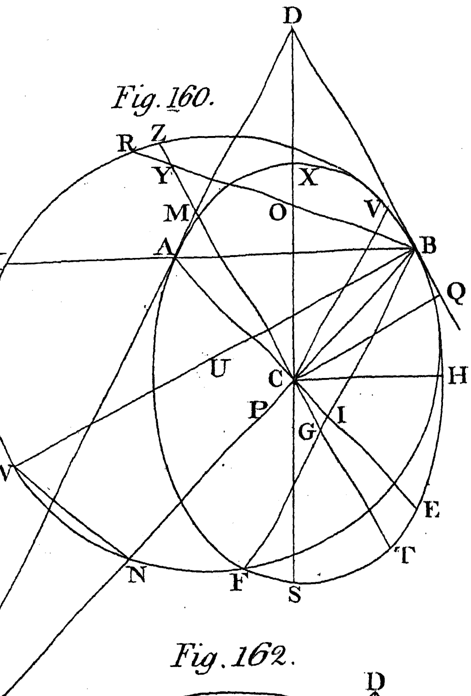


Fig. 164.

Fig

Fig. 163.

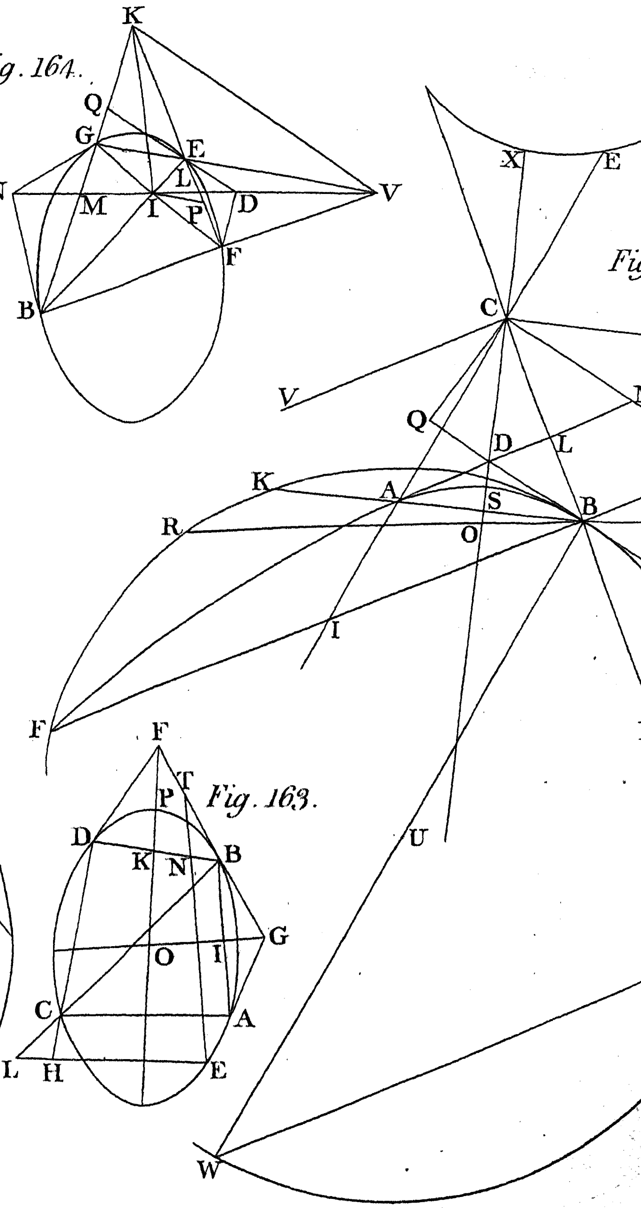
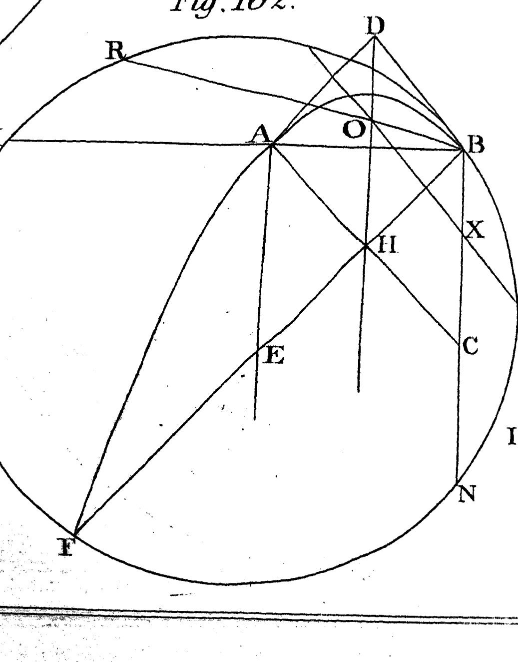


Fig. 162.



that is, by Prop. XIII. Book I. in the ratio compounded of the ratio of the rectangle B G D to the rectangle A G C, and of the ratio of the rectangle A T C to the rectangle B F D. The square of E T therefore (11. v.) is to the square of E F in the ratio compounded of the ratio of the rectangle B G D to the rectangle A G C, and of the ratio of the rectangle A T C to the rectangle B F D. BOOK IV.

Secondly, let the section be an ellipse, of which O is the center, and, the rest being as above, let O a be the semidiameter parallel to B D, O d the semidiameter parallel F T, and O b the semidiameter parallel to A C. Let it be $VX^2 : AT \times TC :: Oa^2 : Ob^2$; and then, as by Prop. V. Book II. $AT \times TC : ET^2 :: Ob^2 : Od^2$, we have Fig. 177.

$$VX^2 : AT \times TC : ET^2$$
$$Oa^2 : Ob^2 : Od^2.$$

Consequently (22. v.) $VX^2 : ET^2 :: Oa^2 : Od^2$; and therefore by Prop. V. Book II. (and 11. v.) $VX^2 : ET^2 :: BF \times FD : EF^2$, and by alternation $ET^2 : EF^2 :: VX^2 : BF \times FD$. Also, by the above, and Prop. V. Book II. (and 11. v.) $VX^2 : AT \times TC : BG \times GD : AG \times GC$. Again, the square of V X is to the rectangle under B F, F D in the ratio compounded of the ratio of the square of V X to the rectangle under A T, T C, that is, by the above, of the rectangle under B G, G D to the rectangle under A G, G C, and of the ratio of the rectangle under A T, T C to the rectangle under B F, F D. The square of E T therefore (11. v.) is to the square of E F in the ratio compounded of the ratio of the rectangle under B G, G D to the rectangle under A G, G C and of the ratio of the rectangle under A T, T C to the rectangle under B F, F D.

Cor. 1. The next remaining as above, if the straight line T F meet in the point H the straight line H L, Fig. 176 177

which

which touches the ſection in the point P and meets T C in L, it may be demonſtrated in the ſame manner that the ſquare of E T is to the ſquare of E H in the ratio compounded of the ratio of the ſquare of P L to the rectangle under A L, L C, and of the ratio of the rectangle under A T, T C to the ſquare of H P.

Cor. 2. If the ſquares of the ſtraight lines M N, R S, T Q, Y Z be equal to the rectangles B G × G D, A G × G C, A T × T C, B F × F D, each to each; then E T will be to E F as the rectangle under M N, T Q to the rectangle under Y Z, R S. For, by the above, the rectangle under A G, G C is to the rectangle under B G, G D as the rectangle under A T, T C to the ſquare of V X; and therefore as the ſquare of R S is to the ſquare of M N, ſo is the ſquare of Q T to the ſquare of V X. Conſequently (22. vi.) as R S is to M N, ſo is T Q to V X, and therefore (16. vi.) the rectangle under R S, V X is equal to the rectangle under M N, T Q. Again, from the above, as the ſquare of E T is to the ſquare of E F, ſo is the ſquare of V X to the rectangle under B F, F D, or the ſquare of Y Z; and therefore, as E T to E F ſo is V X to Y Z, that is (1. vi.) the rectangle under V X, R S to the rectangle under Y Z, R S. Conſequently, on account of the equal rectangles, E T is to E F as the rectangle under M N, T Q to the rectangle under Y Z, R S.

Cor. 3. Hence if any two ſtraight lines A C, B D cut a conic ſection in A, C and B, D, and meet one another in G, and meet in T and F the ſtraight line T F, which touches the ſection, the point of contact may be found. For let E be ſuppoſed to be the point of contact, and as above E T is to E F as the rectangle under a mean proportional between B G, G D and a mean proportional between A T, T C to a rectangle under a mean proportional between B F, F D and a

mean

mean proportional between A G, G C. In the ſame way the point E may be found, if the tangent T F meet in H the ſtraight line H L, which touches the ſection in P, and meets in L the ſtraight line A C, which cuts the ſection in A, C, and meets T F in T.

PROP. XXI. PROB. II.

Four points in the curve of a conic ſection, and a ſtraight line touching the ſection, being given, let it be required to deſcribe the ſection.

Let the four points A, B, D, C in the curve of a conic ſection, and the ſtraight line T F touching the ſection, be given in poſition; it is required to deſcribe the ſection. Fig. 176. 177. 178. 179.

Caſe 1. Let the tangent T F paſs through the point B, and draw A B, B D, D C, C A. Firſt, let the oppoſite ſides A C, B D of the trapezium be parallel, and let K R be drawn biſecting B D in K, and A C in R. Then K R is a diameter, by Cor. 2. Prop. II. Book III. and if it is parallel to the tangent F T, it will be the conjugate diameter to B D, by Cor. 2. Prop. IV. Book II. and the point K will be the center of the ſection. Conſequently, if from the point A a ſtraight line be drawn ordinately applied to the diameter B D, the magnitude of the diameter K R will be determined, by art. 10. and the ſection may then be deſcribed, by art. 8. But if the diameter K R be not parallel to the tangent F T, let them meet in F. Draw F D, and it will touch the ſection, by Cor. 2. Prop. VI. Book III. Through C draw the ſtraight line C I parallel to the tangent F T, and let it meet the tangent F D in E, and B D in G. Let the ſegment E I be ſo taken in C G, that C E may be to E G as E G to E I; and when the point G is without the ſection, let the point C be between E and I. Fig. 178.

Then

BOOK IV.

Then the point I will be in the curve of the section, by Prop. XVII. Book I. and the five points A, B, D, C, I being in the curve, the section may be described as in Prop. XIX.

Fig. 179. Secondly, let the opposite sides of the trapezium be not parallel, and let A B, C D meet in L. Draw A D, C B, and let them meet in P. Through the points P, L draw the straight line P L, and let it meet the tangent T F in F, and draw F D. Then F D will touch the section, by Prop. XVIII. and, as above, a fifth point I may be found in the curve.

Fig. 176. 177. Case 2. If the tangent T F do not pass through a given point, the point of contact may be found, by Cor. 3. Prop. XX. and the section may be described as in Prop. XIX.

PROP. XXII. PROB. III.

Three points being given in the curve of a conic section, and two straight lines touching it being given in position, let it be required to describe the section.

Case 1. If each of the two tangents pass through one of the given points, the points of contact will be given, and a straight line drawn through the third given point in the curve, and parallel to the straight line joining the points of contact, will meet the tangents. The segments of this straight line between the given point in the curve and the tangents will therefore be given; and if they are unequal, the line will be a secant, and the other point in which it cuts the curve may readily be obtained, according to Prop. IV. The solution may then be completed, by Prop. XXI. But if the segments are equal, the line will be a tangent, and in this case call it the third tangent, and draw a straight line from its point of contact to the point of

con-

contact of the firſt tangent. Draw a ſtraight line from the point in which the firſt and third tangents meet to the point of contact of the ſecond tangent. Then, by Prop. XVI. this ſtraight line will be harmonically divided in the point in which the firſt and third tangents meet, the point in which it meets the ſtraight line joining their points of contact, the point of contact of the ſecond tangent, and the other point in which it meets the curve. This other point may therefore be found by the ſecond and third Corollaries in page 155, and the ſolution may be completed as above.

Caſe 2. Let the firſt tangent paſs through one of the given points, but let the ſecond tangent not paſs through either of the other two; and, firſt, let the two tangents be parallel. Through the two other points draw a ſtraight line, and if it be parallel to the tangents, a ſtraight line drawn from the point of contact of the firſt tangent and biſecting this ſecant will meet the ſecond tangent in its point of contact, as is evident from Prop. I. Book II. But if the ſtraight line paſſing through the other two given points be not parallel to the tangents, it will meet them, and then, by Prop. XIII. Book I. the rectangle under its ſegments between the points and the firſt tangent will be to the ſquare of the firſt tangent, as the rectangle under its ſegments between the given points and the ſecond tangent to the ſquare of the ſecond tangent. The point of contact of the ſecond tangent will therefore be obtained. Secondly, let D G be the firſt tangent paſſing through B one of the given points, and let A, C be the other two; and let D K the other tangent not be parallel to D G, but let them meet in D. Draw A C, and let it meet D K in K. Then if A C be parallel to one of the tangents, as in Fig. 180. let K E be a mean proportional between C K, K A, the point E being in K C; and Fig. 180. 181.

and a ſtraight line drawn through B and E will meet D K in its point of contact, by Prop. XVII. Book I. But if the ſtraight line A C be not parallel to either tangent, as in Fig. 181. let it meet D G in G, and D F in K; and let G K be ſo divided, that the rectangle under A K, K C may be to the rectangle under C G, G A as the ſquare of K E to the ſquare of E G. Draw B E, and it will meet the tangent D F in the point of contact, by Cor. 2. Prop. XVII. Book I.

Fig. 182. Caſe 3. Let neither of the tangents K F or D G paſs through a given point, and let A, B, C be the given points. Draw A C, and let it meet the tangent D G in D, and the tangent K F in K. Let K D be ſo divided in E, that the rectangle under A K, K C may be to the rectangle under C D, D A as the ſquare of K E to the ſquare of E D, and the ſtraight line joining the points of contact will paſs through E, by Cor. 2. Prop. XVII. Book I. Again, draw C B, and let it meet the tangent D G in L, and the tangent K F in I. Let L I be ſo divided in M, that the rectangle under C L, L B may be to the rectangle under B I, I C as the ſquare of L M to the ſquare of M I, and the ſtraight line joining the points of contact will paſs through M, by the ſame as above. Conſequently the ſtraight line M E will meet the tangents in the points of contact.

In every caſe, therefore, a ſection may be deſcribed, by Prop. XIX. or Prop. XXI.

Cor. If two tangents be given in poſition, and alſo two points in the curve of a conic ſection, but without the tangents, the point may be found in which the ſecant, paſſing through the given points, meets the ſtraight line joining the points of contact, provided the ſecant be not parallel to the ſtraight line joining the points of contact. For if the tangents and ſecant be parallel to one another, the point in which the ſecant is

is bisected will be the point required, as stated in the second case. In other cases the Cor. is evident from the above; for the tangents D G, D K being given in position, and the secant passing through the given points A, C, as in Fig. 180, 181, 182. the point E was ascertained.

PROP. XXIII.

If three straight lines touch a conic section, a straight line drawn through the point in which the first and second tangents meet one another will be harmonically divided in this point of concourse, in the point in which it meets the third tangent, and in the points in which it meets straight lines drawn from the first and second points of contact through the third point of contact.

Let the straight lines E F, E G, G H touch the conic section in the points F, C, H, the straight line E N, drawn through the point E, in which the first tangent E F and the second E G meet one another, is harmonically divided in E, in the point R in which it meets the third tangent G H, and in the points P, N in which it meets the straight lines F H, C H, drawn from the first and second points of contact F, C through the third point of contact H. Fig. 189.

For let E N meet the curve of the section in the points S, I; and then, by Cor. 2. Prop. XVII. Book I, $SE \times EI : SR \times SI :: EP^2 : PR^2 :: EN^2 : NR^2$. Consequently (22. i.) $EP : PR :: EN : NR$.

PROP. XXIV. PROB. IV.

Two points being given in the curve of a conic section, and three straight lines being given in position and touching the curve, let it be required to describe the section.

 The

BOOK IV.

The two points A, B being given in the curve of a conic ſection, and the three ſtraight lines C D, E F, G H being given in poſition and touching the ſection, let it be required to deſcribe the ſection.

Fig. 183. Caſe 1. Let C D, G H two of the given tangents paſs through the two given points A, B, and let them meet the other given tangent E F in D and G. If the tangent E F be parallel to A B, the ſtraight line joining the points of contact, and G D be biſected in E, the point E will be that in which G D touches the ſection, by Prop. IV. But if G D, A B be not parallel, let them meet in F, and let G D be ſo divided that D F may be to F G as D E to E G, and E will be the point in which G D touches the ſection, by Prop. XV. Three points will therefore be obtained in the curve, and conſequently the ſolution may be completed, by Prop. XXII.

Fig. 184. 185. Caſe 2. Let G H one of the given tangents paſs through B one of the given points, and meet the given tangent E F in G. Let the ſtraight line A B be drawn, and let it meet the given tangent C D in D; and if A B be parallel to E F, let the ſegment D L be taken in B D a mean proportional between B D, D A, and the ſtraight line joining the points in which E F, C D touch the ſection, will paſs through the point L, by Prop. XVII. Book I. Draw G L, and let it meet the tangent C D in K; and if the tangents G H, C D be parallel, let L G be to L K as B G to K C, and C will be the point in which C D touches the ſection. For let G K meet the curve of the ſection in P and N, and the rectangle under P G, G N is to the rectangle under N K, K P as the ſquare of G L to the ſquare of L K, by Prop. XVII. Book I. and the ſquare of B G is to the ſquare of C D, the ſegment between the point K and the point of contact, in the ſame ratio, by Prop. XIII. Book I. If the ſtraight line A B be not parallel to the tangent E F, the point

point L may be found by Cor. 2. Prop. XVII. Book I. and the tangents G H, C D being parallel, the point of contact C may be found as above.

Fig. 185. But if the tangent G H meet the tangent E F in G, and the tangent C D in H, let G H be so divided in N, that H B may be to B G as H N to G N, and the straight line joining the points of contact of the tangents E F, C D will pass through N, by Prop. XV. Again, let the straight line A B meet the tangent E F in P, and the tangent C D in D, and as the rectangle under A P, P B is to the rectangle under B D, D A, so let the square of P L be to the square of D L, and the straight line joining the points of contact of the tangents E F, C D will pass through L, by Cor. 2. Prop. XVII. Book I. Let the straight line L N therefore be drawn, and it will meet the tangents E F, C D in the points of contact. If the straight line A B be parallel to the tangent F G, the point L may be found, by Prop. XVII. Book I. as above.

Fig. 186. Case 3. Let the points A, B be without any one of the tangents, and let the straight lines A B, E F, G H be parallel. Let the tangent C D meet the tangent G H in G, and the tangent E F in E; and let the straight line A B meet the tangent C D in D. In A D let the segment D L be taken a mean proportional between A D, D B, and the straight line joining the points of contact of the tangents C D, E F will pass through the point L, by Prop. XVII. Book I. Let the straight line G L N be drawn; let it meet the tangent E F in K, and let G L be to L K as G N to N K, and the straight line joining the points of contact of the tangents G H, E F will pass through N, by Prop. XXIII. Again, let A B be bisected in P, and the straight line joining the points of contact of the tangents G H, E F will pass through P, by Prop. I. Book II. Consequently if the straight line

BOOK IV. N P be drawn, it will meet the tangents in the points of contact.

Fig. 187. Case 4. Let the tangent C D meet the tangent G H in G and E F in E, and let the straight line A B be parallel to the tangent C D, and meet the tangent G H in K, and the tangent E F in M. In the straight line K M let the segment M P be taken a mean proportional between A M, M B; and let the segment K L be a mean proportional between B K, K A; and the straight line joining the points of contact of the tangents C D, E F will pass through the point P, but the straight line joining the points of contact of the tangents C D, G H will pass through the point L, by Prop. XVII. Book I. Let the straight line G P be drawn, and meet the tangent E F in I, and as G P to P I so let G N be to N I. Again, let L E be drawn, and let it meet the tangent G H in Q, and let Q L be to L E as Q R to R E. Then, by Prop. XXIII. the straight line N R passes through the points of contact of the tangents E F, G H.

Fig. 188. Case 5. Let the straight line A B be not parallel to any one of the tangents G H, D C, E F, and let it meet the tangents D C, E F in E, the point in which they meet one another. Let the straight line A B be divided in the point L, so that B E may be to E A as B L to L A, and the straight line joining the points of contact of the tangents D C, E F will pass through the point L, by Prop. XVI. Let the straight line G L be drawn, and meet the tangent E F in K, and in the straight line G L let the segment G N be so taken, that G L may be to L K as G N to N K; and, by Prop. XXIII. the straight line joining the points of contact of the tangents E F, G H passes through the point N. Again, let the straight line A B meet the tangent G H in M, and in A B let the segment A R be so taken that the rectangle A E B may be to the rectangle A M B as the square of E R to the square

quare of R M, and the ſtraight line joining the points of contact of the tangents E F, G H will paſs through the point R, by Prop. XVII. Book I. Conſequently the ſtraight line N R will meet the tangents E F, G H in the points of contact.

Caſe 6. Let the tangent C D meet the tangent G H in G, and the tangent E F in E. Let the ſtraight line A B be drawn, and let it meet the tangent G H in M, the tangent E F in K, and the tangent C D in D. Let the ſtraight line A B be ſo divided in L and N, that the rectangle under A K, K B may be to the rectangle under B M, M A as the ſquare of K L to the ſquare of M L; and as the rectangle under A D, D B to the rectangle under B M, M A, ſo let the ſquare of D N be to the ſquare of N M. Then the ſtraight line joining the points of contact of the tangents E F, G H will paſs through the point L, but the ſtraight line joining the points of contact of the tangents C D, G H will paſs through N, by Prop. XVII. Book I. Let the ſtraight line N E be drawn, and meet the tangent G H in R; and let N E be ſo divided in P, that E N may be to N R as E P to P R; and, by Prop. XXIII. the ſtraight line joining the points of contact of the tangents E F, G H paſſes through P. Conſequently if the ſtraight line L P be drawn, it will meet the tangents E F, G H in the points of contact. Fig. 189.

In every caſe therefore the ſection may be deſcribed by Prop. XIX. as five points may be eaſily found.

PROP. XXV.

If the four ſtraight lines A E, E G, G H, H D *touch a conic ſection in* A, B, C, D, *and meet one another in* E, G, F, H; *and if the ſtraight lines* A C, B D *be drawn joining the oppoſite points* A, C *and* B, D *and meeting one another; the ſtraight line* F G *drawn through the oppo-* Fig. 174.

BOOK IV.

ſite points in which the tangents meet one another will paſs through the point in which the ſtraight lines A C, B D *meet one another.*

For let the ſtraight line F G meet the ſection or oppoſite ſections in N, M and the ſtraight line A C in I, and, by Cor. 2. Prop. XVII. Book I. the rectangle under N G, G M is to the rectangle under N F, F M as the ſquare of G I to the ſquare of F I. But, if it be poſſible, let the ſtraight line F G meet the ſtraight line B D in P, and, as above, the rectangle under N G, G M is to the rectangle under N F, F M as the ſquare of P G to the ſquare of F P. Conſequently (11. v.) as the ſquare of G I to the ſquare of F I, ſo is the ſquare of P G to the ſquare of F P; and therefore (22. vi.) as G I to F I ſo is P G to F P. Conſequently (18. v.) as G I to F G ſo is G P to F G, and therefore (14. v.) the ſtraight lines G I, G P are equal; which is abſurd.

The reſt remaining, if the ſtraight line F G be a diameter of a parabola, or parallel to an aſymptote of an hyperbola, the ſquare of N I, and alſo the ſquare of N P, will be equal to the rectangle under G N, N F, by Prop. XXIII. Book III. and therefore G I, G P are equal; which is abſurd.

In any caſe therefore the ſtraight lines A C, B D, F G meet one another in I, and if the ſtraight line E H be drawn joining the remaining oppoſite points, in which the tangents meet one another, it will paſs through the point I, for the ſame reaſons as above.

Cor. Hence, if four ſtraight lines A E, E G, G H, H D touch a conic ſection and meet one other in E, G, F, H, and if the ſtraight lines F G, E H joining the oppoſite points meet one another in I; the ſtraight lines A C, B D joining the oppoſite points of contact will paſs through the point I.

PROP.

PROP. XXVI. PROB. V.

The point A *being given in the curve of a conic section, and the four straight lines* E F, E G, G H, H D *touching the section being given in position, let it be required to describe the section.* Fig. 174. 175.

Let the tangents meet one another in E, F, G, H, and let the straight lines F G, E H be drawn, joining the opposite points F, G and E, H, and meeting one another in I. Let the straight line A I be drawn, and let it meet the opposite tangent G H in C, and if the point A be in one of the tangents, the straight line G H will touch the section in C, as is evident from Prop. XXV. But if the point A be not in one of the tangents, let A C meet E F in K, and let the straight line K C be so divided in R that the square of K I may be to the square of I C as the rectangle A K R to the rectangle R C A; and as by the last Prop. the straight line joining the points of contact of E F, G H passes through I, the point R will be in the curve of the section, by Cor. 2. Prop. XVII. Book I. Consequently in any case the section may be described, by Prop. XXIV.

PROP. XXVII. PROB. VI.

Five straight lines being given in position and touching a conic section, let it be required to find the points in which they touch the section.

Let the straight lines A B, B C, C D, D E, E A touch a conic section, and let it be required to find the points of contact in them. Fig. 190.

Let A B C D E A be the quinquelateral figure contained by the tangents, and let A B be called the first side, B C the second, &c. and let F B C D be the quadrilateral figure contained under the four first sides, and draw the diagonals B D, F C meeting one another in M.

The

BOOK IV.

The first side A B of the quinquelateral figure being now omitted, let I C D E be a quadrilateral contained by the others, and let I D, C E the diagonals be drawn meeting one another in N. Then M N being drawn, it will pass through the points in which the second side B C and the fourth D E touch the section, by Cor. Prop. XXV. In the same manner the points may be found in which A B, C D, A E touch the section, and therefore the section may be described, by Prop. XIX.

PROP. XXVIII.

Fig. 191. *Let* E D *be an equilateral hyperbola, of which* A F, A C *are the asymptotes, and let it cut in the point* D *the curve of the parabola* A D, *of which* A F *is the axis, and the segment* A F *equal to the parameter of the axis; let there be drawn to the curve of the hyperbola the straight line* F E *parallel to the asymptote* A C, *and from the point* D, *in which the curves of the hyperbola and parabola cut one another, let there be drawn to the asymptote* A F *the straight line* D B *parallel to the asymptote* A C; *then will the straight lines* B D, A B *be two mean proportionals between* A F, F E.

For as E D is an equilateral hyperbola, the angle A F E is a right one, by Prop. XVI. Book III. (and 29. i.) The straight line D B is therefore an ordinate to the axis of the parabola, and, by Prop. III. Book III. (and 17. vi.) A F : B D : : B D : A B. Again, by Cor. 2. Prop. XVII. Book III. A F : A B : : B D : F E, and therefore by alternation A F : B D : : A B : F E. Consequently (11. v.) A F : B D : : B D : A B, and B D : A B : : A B : F E.

Cor. Hence if two straight lines as A F, F E be given, two mean proportionals may be found between them. For let the two straight lines A F, F E be at right angles

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Fig. 165.

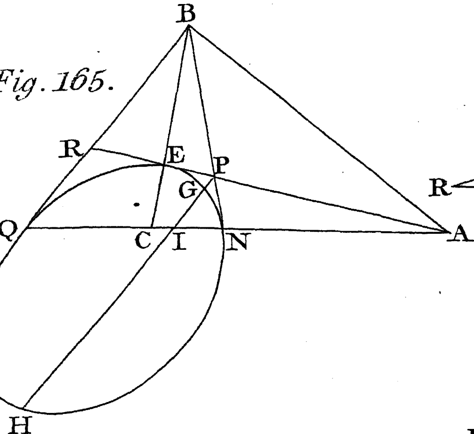


Fig. 166.

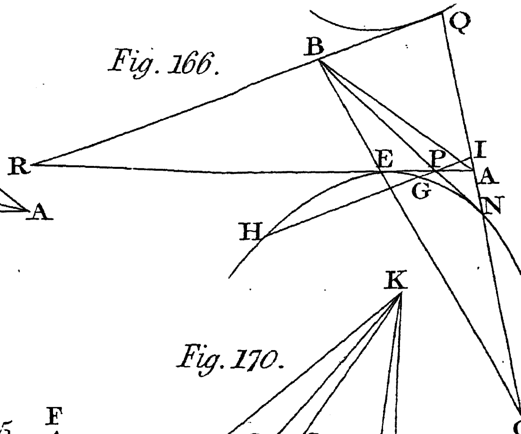


Fig. 17

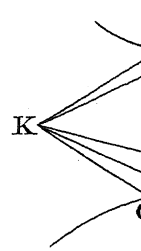


Fig. 170.

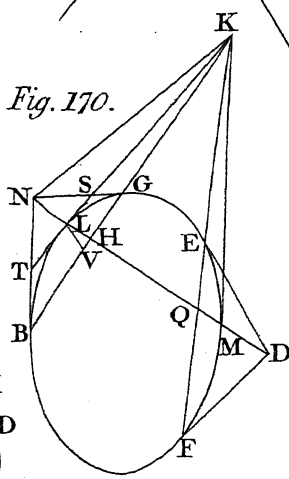


Fig. 169.

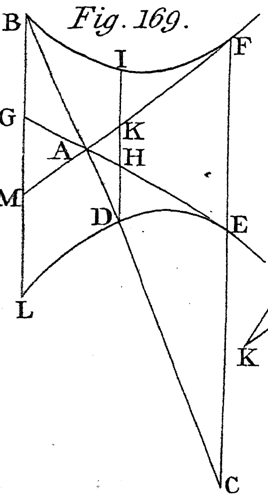
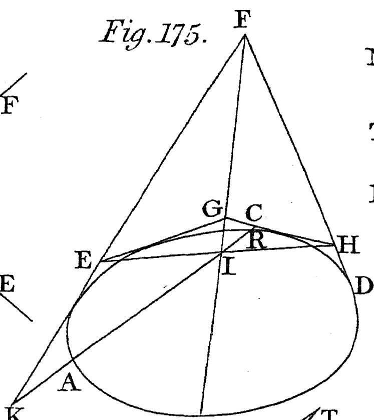


Fig. 175.



Fig

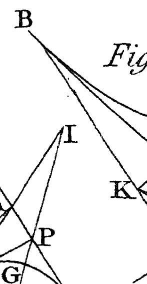
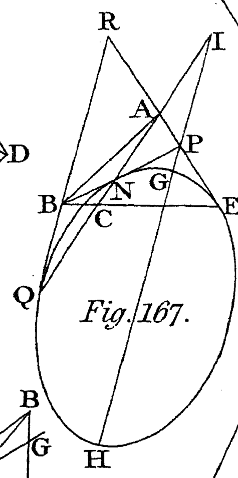


Fig. 167.



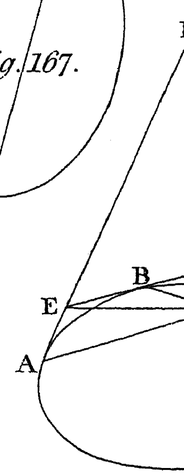


Fig. 168.

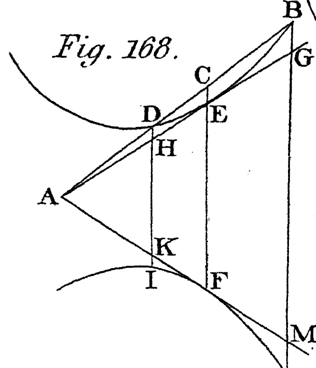
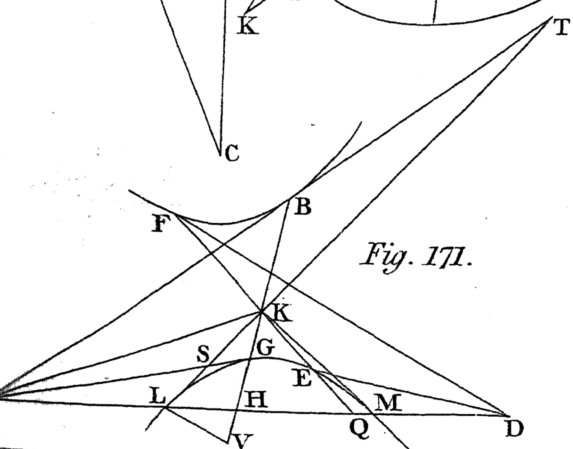


Fig. 171.



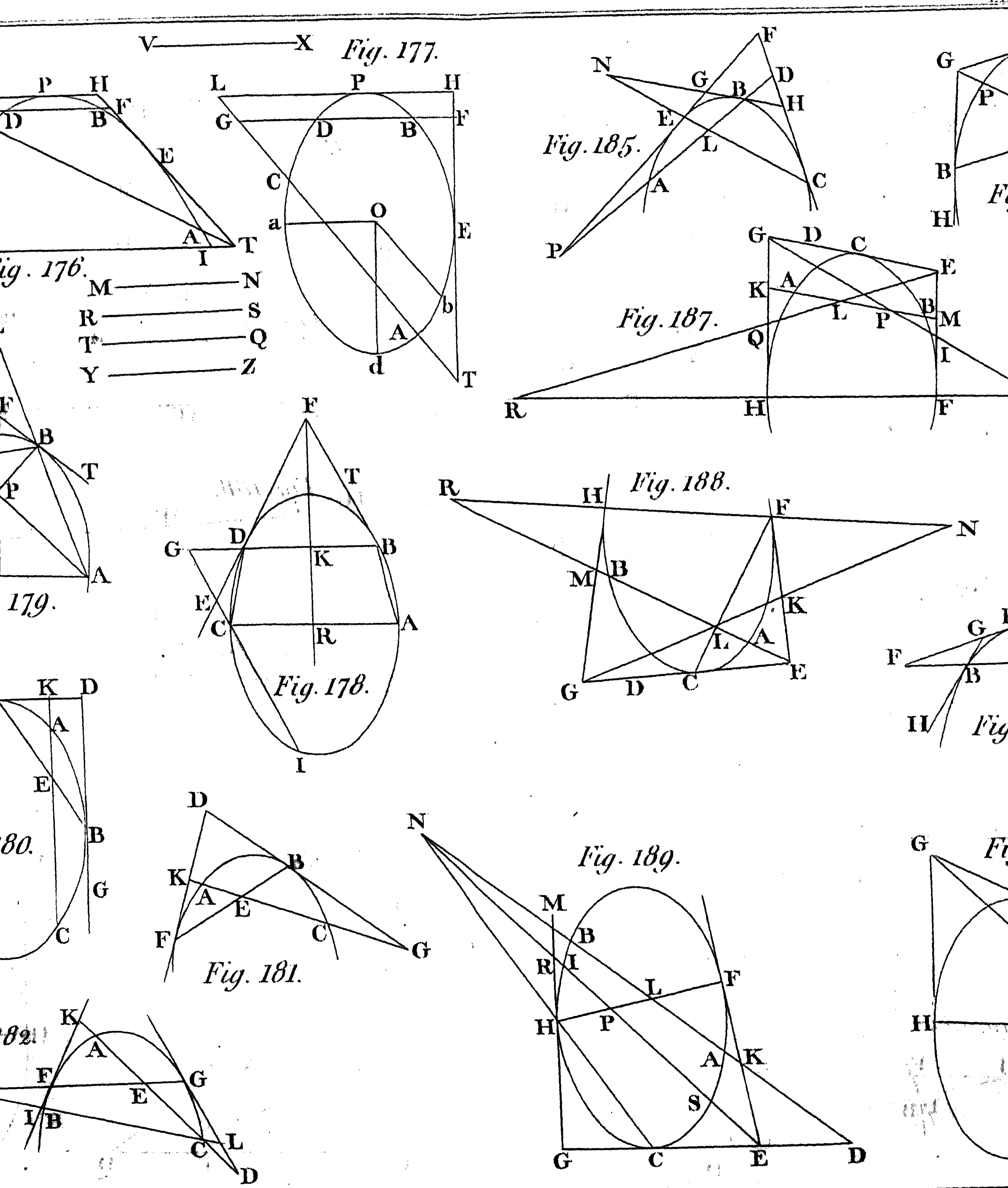


Fig. 177.
Fig. 185.
Fig. 187.
Fig. 178.
Fig. 188.
Fig. 181.
Fig. 189.

gles to one another, and let the parallelogram A F E G be completed. Let the parabola A D be described, of which A F is the axis, and the segment A F equal to its parameter. Again, let an equilateral hyperbola be described through the point E, of which A F, A G are the asymptotes, and let its curve cut the curve of the parabola in D. Let D B be drawn to A F and parallel to A G, and let D C be drawn to A G parallel to A F. Then the straight lines B D, A B will be two mean proportionals between A F, F E.

PROP. XXIX.

Let A E *be a parabola, of which* A D *is the axis, and* A B *a segment in it equal to half its parameter; let the straight line* B G *be perpendicular to the axis, and draw* A G; *with the center* G *and distance* G A *describe the circle* A C E *cutting the axis in the point* C *and the curve of the parabola in* E, *and let* E D *be drawn an ordinate to the axis; the straight lines* E D, A D *will be two mean proportionals between* A C *and a straight line equal to the double of* G B. Fig. 192.

For, by the construction, (and 3. iii.) the straight line A C is equal to the parameter of the axis, and therefore, by Prop. III. Book III. the square of D E is equal to the rectangle under A C, A D. Let D E meet the circumference of the circle again in F, and let the segment F H be equal to the segment D E. Then the rectangle E D F will be equal to the rectangle A D C, (35. iii.) and therefore the square of D E together with the rectangle E D F are equal to the rectangles D A C, A D C together, that is, (2. ii.) to the square of A D. But the square of D E together with the rectangle E D F is equal to the rectangle under D E, and a straight line equal to the sum of E D, D F (1. ii.); and therefore

BOOK IV. fore the ſquare of D E together with the rectangle E D F is equal to the rectangle under D E, H D. Conſequently (17. vi.) D E : A D : : A D : H D; and by the above A C : D E : : D E : A D. But D H is double of G B; for let G I be drawn parallel to A D, and let it meet D H in I. Then G B, I D (34. i.) are equal to one another, as are alſo (3. iii.) E I, I F to one another, and therefore H I is equal to I D. The Propoſition is therefore evident.

Cor. Hence, by means of a parabola and a circle, a method is evident of finding two mean proportionals between two given ſtraight lines.

PROP. XXX.

Fig. 193. *From any point* B *in the curve of the equilateral hyperbola* B E *let the ſtraight lines* B A, B D *be drawn to the aſymptotes* C A, C D, *and let* B A *be parallel to* C D *and* B D *parallel to* C A, *and let* A D *the diameter of the parallelogram be drawn; with the center* B *and a diſtance equal to the double of* A D *let a circle be deſcribed, and let it meet the curve of the hyperbola in* E; *from* E *draw* E F *to the aſymptote* C D *and parallel to* C A; *then,* A F *being drawn, the angle* B A F *will be a third part of the angle* B A D.

For let A F meet B D in G. Biſect D F in K, and draw K I parallel to B D, and let it meet A F in the point I. Draw D I. Then as the hyperbola is equilateral, the angle A C D is a right one, and therefore (29. i.) each of the angles F K I, D K I is a right one, and (4. i.) F I, I D are equal. But, on account of the equals F K, K D and the parallels K I, D G, F I is equal to I G. Again, (15. and 29. i.) the triangles A B G, F C A are equiangular, and therefore (4. vi.) A B : B G : : C F : C A, and (16. vi.) the rectangle under B G, C F is equal to

to the rectangle under B A, A C. But, by Prop. XVII. Book III. the rectangle under B A, A C is equal to the rectangle under E F, C F, and therefore the rectangle under B G, C F is equal to the rectangle under E F, C F. Consequently B G is equal to E F, and therefore (33. i.) B E is equal to G F; and therefore, by the construction, F I, I D, D A are equal. The angles D F I, F D I are therefore equal to one another, as are also the angles D I A, D A I to one another. The angle D A I is therefore equal to the double (32. i.) of D F I, or of its equal the angle B A G. Consequently the angle B A G is equal to a third part of the angle B A D.

Cor. Hence, by means of an equilateral hyperbola and its asymptotes, an angle may be divided into three equal parts.

A

TREATISE

ON THE

PRIMARY PROPERTIES

OF

CONCHOIDS, THE CISSOID, THE QUADRATRIX, CYCLOIDS, THE LOGARITHMIC CURVE,

AND

THE LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

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AND THE

OGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

SECTION I.

Of Conchoids.

DEFINITIONS.

I.

IF the fixed point P be without the ſtraight line T R X, and if the ſtraight line D L of indefinite length aſs through P, and D, A be two fixed points in D L; hen if the ſtraight line D L, always paſſing through P, e moved in ſuch a manner that the point A is always i T R X, and the point D deſcribe the curve D V G, the urve D V G is called a *Conchoid*. Plate XXVI. Fig. 1. 2. 3.

II.

The fixed point P is called the *Pole* or *Center* of the onchoid, and the ſtraight line T R X is called the *Di-ectrix* of the conchoid.

SECT. I.

III.

The ſtraight line P R V perpendicular to the directrix, meeting it in R and the curve in V, is called the *Axis* of the conchoid; and the point V is called the *Vertex* of the axis.

IV.

If the directrix be between the curve and the pole, as in Fig. 1. the curve is called the *Superior Conchoid*, or the *Conchoid of Nichomedes*.

V.

If the curve be between the directrix and the pole, and the ſegment R V of the axis be leſs than the ſegment P R, as in Fig. 2. the curve is called the *Inferior Conchoid*.

VI.

If the curve and pole be on the ſame ſide of the directrix, and the ſegment V R of the axis be greater than the ſegment P R, as in Fig. 3. the curve D P V G is called the *Nodated Conchoid*.

Corollary to the preceding Definitions. In any one of the three conchoids if a ſtraight line paſs through the pole, and cut the directrix and curve, its ſegment intercepted between the directrix and the curve will be equal to the ſegment R V of the axis, between the directrix and curve. For the points D, A being fixed in the ſtraight line D L, the magnitude of the ſegment D A is the ſame in every poſition of the moving line D L; and when D L falls upon the axis the deſcribing point D coincides with V, and the point A coincides with R.

VII.

A ſtraight line drawn from any point in the conchoid perpendicular to the axis is called an *Ordinate* to the axis.

PROP.

PROP. I.

SECT. I.

The conchoid and its directrix being produced, on either side of the axis, continually approach nearer and nearer to one another, but never meet.

For, the rest remaining as in the Definitions of each of the conchoids, let the straight line D B be perpendicular to the directrix T X. Then P R, D B being perpendicular to the directrix T X, the triangles (15. and 29. i.) A P R, A D B are equiangular; and therefore (4. vi.) A P : P R : : A D : D B. Consequently, as by the Cor. to the Definitions A D is equal to R V, the rectangle (16. vi.) under A P, D B is equal to the rectangle under P R, R V. But D B is the distance of the curve at the point D from the directrix; and it is evident that A P increases as the distance of D from the axis increases. The distance of the describing point D, therefore, from the directrix must decrease as D recedes from the vertex, as P R, R V are constant; and as the rectangle under A P, D B is a constant magnitude, the points D and B cannot coincide. Hence the Proposition is evident. Fig. 1. 2. 3.

Cor. The directrix is also an asymptote to the conchoid.

PROP. II.

If an ordinate be drawn from any point in either of the conchoids to the axis, a straight line drawn from the pole to the same point in the curve will be a fourth proportional to the distance of the ordinate from the directrix, the distance of the vertex from the directrix, and the distance of the ordinate from the pole.

The rest remaining as in the preceding Proposition, and Fig. 1. 2. 3.

SECT. I.

and the fix firft Definitions, let D E be an ordinate t the axis according to the feventh Definition, and let : meet the axis in E; and then R E is to R V as P E t P D.

For B E is a parallelogram, and therefore (34. i.) R: is equal to B D, and, by the Cor. to the Definitions A D is equal to R V. Confequently R E : R V : : B D A D. But the triangles (29. i.) B D A, E P D are equi angular, and therefore (4. vi.) B D : A D : : P E : P D Confequently (11. v.) R E : R V : : P E : P D.

Cor. 1. The reft remaining as above, with R as a center and R V as a diftance defcribe V I C a quadrant of a circle, and let it cut D E, or D E produced, in I, and draw I R. Then I R is equal to R V, and therefore by the above R E : I R : : P E : P D. In the fuperior conchoid the angle at E is common to the two triangles R E I, P E D, and in the other two conchoids, the angle at E in the triangle R E I is equal to the angle at E in the triangle P E D, each of them being a right angle. Confequently (7. vi.) the triangles R E I, P E D are equiangular, and therefore (4. vi.) P E : E D : : R E : I E.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be eafily deduced. For in each of them put R V $= a$, P R $= b$, R E $= x$, and D E $= y$.

Fig. 1. 1. In the fuperior conchoid (47. i.) I E $= \sqrt{a^2 - x^2}$, as I R, R V are equal; and P E $= b + x$. Confequently $b + x : y :: x : \frac{x\,y}{b+x} =$ I E $= \sqrt{a^2 - x^2}$; and $y = \frac{\overline{b+x} \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid I E $= \sqrt{a^2 - x^2}$, for the fame reafons as above; and P E $= b - x$. Proceeding there-

therefore as in the laſt article, the equation of the inferior conchoid is $y = \dfrac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. SECT. I.

3. In the nodated conchoid I E $= \sqrt{a^2 - x^2}$, as in the other two, and P E $= b - x$; and therefore the equation of the nodated conchoid is alſo $y = \dfrac{\overline{b-x} \times \sqrt{a^2 - x^2}}{x}$. Fig. 3.

4. The foregoing equations being cleared of the ſurd, the equation of the ſuperior conchoid becomes $x^4 + 2\,b\,x^3 + b^2\,x^2 - a^2\,x^2 + y^2\,x^2 - 2\,a^2\,b\,x = a^2\,b^2$; and the equation of each of the other two becomes $x^4 - 2\,b\,x^3 + b^2\,x^2 - a^2\,x^2 + y^2\,x^2 + 2\,a^2\,b\,x = a^2\,b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoid, publiſhed an account of an inſtrument for the deſcription of the curve *, conſtructed upon the principles ſtated in the firſt Definition, of which the following is the ſubſtance.

Let T X, P R be two rulers of wood or metal, fixed at right angles to one another at R; and let them be of indefinite length, and have each a ſmooth groove to a convenient extent, as repreſented. Let D L be another ruler of wood or metal, of indefinite length, and let it alſo have a ſmooth groove of a convenient extent, as repreſented. Let P be a pin which may be fixed in the ruler P R at any requiſite diſtance from the point R. Let A be a pin which may be fixed in the ruler D L at any requiſite diſtance from the extremity D. Let the ruler D L be adapted to the other two by means of the pins A, P in ſuch a manner, that the pin A, fixed in D L, may ſlide ſmoothly in the groove in T X, and that the groove in D L may always embrace, Fig. 4.

* See the Oxford edition of Archimedes, page 147.

SECT. I.

and the ſix firſt Definitions, let D E be an ordinate to the axis according to the ſeventh Definition, and let it meet the axis in E; and then R E is to R V as P E to P D.

For B E is a parallelogram, and therefore (34. i.) R E is equal to B D, and, by the Cor. to the Definitions, A D is equal to R V. Conſequently R E : R V : : B D : A D. But the triangles (29. i.) B D A, E P D are equiangular, and therefore (4. vi.) B D : A D : : P E : P D. Conſequently (11. v.) R E : R V : : P E : P D.

Cor. 1. The reſt remaining as above, with R as a center and R V as a diſtance deſcribe V I C a quadrant of a circle, and let it cut D E, or D E produced, in I, and draw I R. Then I R is equal to R V, and therefore by the above R E : I R : : P E : P D. In the ſuperior conchoid the angle at E is common to the two triangles R E I, P E D, and in the other two conchoids, the angle at E in the triangle R E I is equal to the angle at E in the triangle P E D, each of them being a right angle. Conſequently (7. vi.) the triangles R E I, P E D are equiangular, and therefore (4. vi.) P E : E D : : R E : I E.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be eaſily deduced. For in each of them put R V $= a$, P R $= b$, R E $= x$, and D E $= y$.

Fig. 1. 1. In the ſuperior conchoid (47. i.) I E $= \sqrt{a^2 - x^2}$, as I R, R V are equal; and P E $= b + x$. Conſequently $b + x : y : : x : \frac{xy}{b+x} =$ I E $= \sqrt{a^2 - x^2}$; and $y = \frac{\overline{b+x} \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid I E $= \sqrt{a^2 - x^2}$, for the ſame reaſons as above; and P E $= b - x$. Proceeding there-

SECT. I.

therefore as in the laſt article, the equation of the inferior conchoid is $y = \frac{\overline{b-x} \times \sqrt{a^2-x^2}}{x}$.

Fig. 3.

3. In the nodated conchoid $IE = \sqrt{a^2-x^2}$, as in the other two, and $PE = b - x$; and therefore the equation of the nodated conchoid is alſo $y = \frac{\overline{b-x} \times \sqrt{a^2-x^2}}{x}$.

4. The foregoing equations being cleared of the ſurd, the equation of the ſuperior conchoid becomes $x^4 + 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 - 2a^2bx = a^2b^2$; and the equation of each of the other two becomes $x^4 - 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 + 2a^2bx = a^2b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoid, publiſhed an account of an inſtrument for the deſcription of the curve *, conſtructed upon the principles ſtated in the firſt Definition, of which the following is the ſubſtance.

Fig. 4.

Let TX, PR be two rulers of wood or metal, fixed at right angles to one another at R; and let them be of indefinite length, and have each a ſmooth groove to a convenient extent, as repreſented. Let DL be another ruler of wood or metal, of indefinite length, and let it alſo have a ſmooth groove of a convenient extent, as repreſented. Let P be a pin which may be fixed in the ruler PR at any requiſite diſtance from the point R. Let A be a pin which may be fixed in the ruler DL at any requiſite diſtance from the extremity D. Let the ruler DL be adapted to the other two by means of the pins A, P in ſuch a manner, that the pin A, fixed in DL, may ſlide ſmoothly in the groove in TX, and that the groove in DL may always embrace,

* See the Oxford edition of Archimedes, page 147.

SECT. I.

but slide smoothly over the pin P, fixed in P R. Then if a pencil or pen be attached to the fixed point D in D L, it will trace out the superior, inferior, or nodated conchoid, according as the conditions stated in the fourth, fifth, or sixth Definition, are attended to in adjusting the instrument.

PROP. III. PROB. I.

Two straight lines being given, let it be required to find two means in continued proportion between them by a conchoid.

Fig. 5. Let L G, L A be two given straight lines; it is required to find two means in continued proportion between them.

Let A L, L G be at right angles to one another, and complete the parallelogram A L G B. Bisect A B in D, and B G in E. Draw L D, and, being produced, let it meet G B produced in H. Draw E C perpendicular to B G, and let it meet G C equal to A D, or D B, in C. Draw H C, and G F parallel to it. With C as a pole, G F as a directrix, and a distance equal to A D or G C between the directrix and vertex, let a conchoid be described, and let it cut H G K in K; and F K will be equal to A D or G C, by the Cor. to the Definitions. Draw K L, and, being produced, let it meet B A in M. Then will G K, M A be the two mean proportionals required.

For (6. ii.) the rectangle under B K, K G, together with the square of E G, is equal to the square of E K; and therefore, by adding the square of E C to these equals, the rectangle under B K, K G, together with the square of C G, (47. i.) is equal to the square of C K. By similar triangles M A : A B : : M L : L K : : B G : G K; and therefore (16. vi.) M A × G K is equal to A B × B G.

$\times$ B G. Again, by similar triangles, L G or A B : D B :: G H : B H, and as A B is double of D B, H G is double of B G, and therefore A D $\times$ H G is equal to A B $\times$ B G. Consequently (16. vi.) M A : A D :: H G : G K :: C F : F K; and (18. v.) M D : A D :: C K : F K. But A D is equal to F K, by construction, and therefore (14. v.) M D is equal to C K, and M D^2 is equal to C K^2. By the above therefore the rectangle under B K, K G, together with the square of C G, is equal to the square of M D, which (6. ii.) is equal to the rectangle under B M, M A together with the square of A D, or its equal C G. Consequently B K $\times$ K G is equal to B M $\times$ M A, and (16. vi.) B M : B K :: G K : M A. But by similar triangles B M : B K :: G L : G K :: M A : A L. Consequently (11. v.) G L : G K :: G K : M A :: M A : A L; and therefore G K, M A are two means in continued proportion between the given straight lines A L, L G.

PROP. IV. PROB. II.

An angle being given, let it be required to divide it into three equal parts, and by means of a conchoid.

Let A C B be a given angle, it is required to trisect it, or divide it into three equal parts. Fig. 6, 7.

With C as a center, and any convenient distance C A, describe the circle A B F, and produce the diameter B F indefinitely. With A as a pole, B F as a directrix, and a distance between the directrix and vertex equal to C A, let a conchoid be described, and let it cut the circumference of the circle in E. Draw A E, and let it meet B F in D, and the angle A D C is a third part of the given angle A C B.

For C E being drawn, E D C is an isosceles triangle, and (5. i.) the angles E D C, E C D are equal, as are

SECT. I.

also the angles C E A, C A E. In Fig. 6. the angle A E C (32. i.) therefore is double of the angle A D C, and consequently the angle C A D is double of the angle A D C. But the angle A C B (32. i.) is equal to the angles C A D, A D C together, and therefore the angle A D C is a third part of the angle A C B. In Fig. 7. the angle E A C, (32. i.) and therefore its equal A E C, is equal to the angles E D C, A C D together; and the angle D A C is equal to the angles A E C, A C E together. Consequently the angle D A C is equal to the angles E D C, A C D, A C E taken together; that is, the angle D A C is equal to the angles E D C, E C D together, or to the double of the angle A D C. Consequently, as the angle A C B (32. i.) is equal to the angles D A C, A D C together, the angle A D C is a third part of the angle A C B.

SECTION

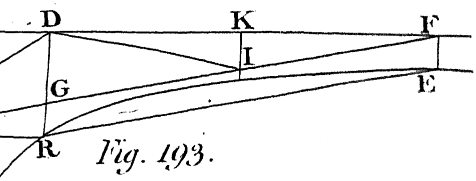


Fig. 193.

Fig. 4.

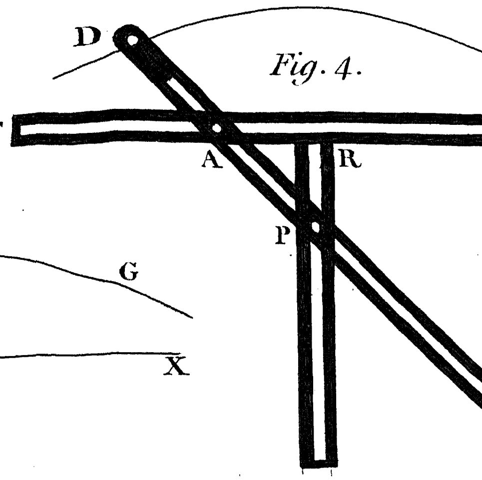


Fig. 1.

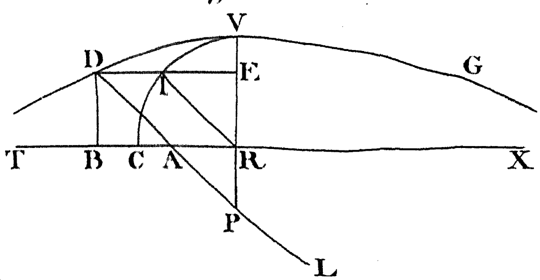
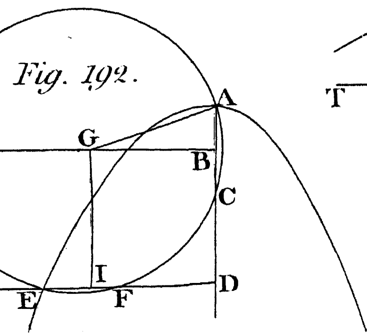


Fig. 192.



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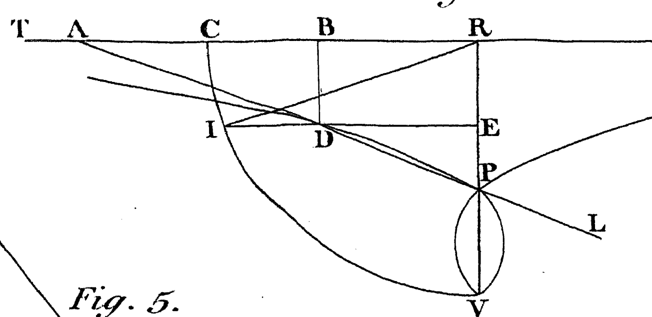


Fig. 190.

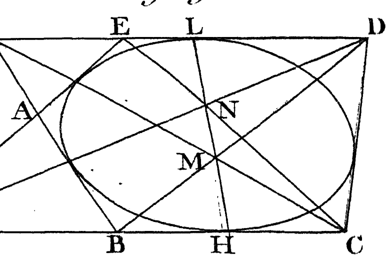


Fig. 5.

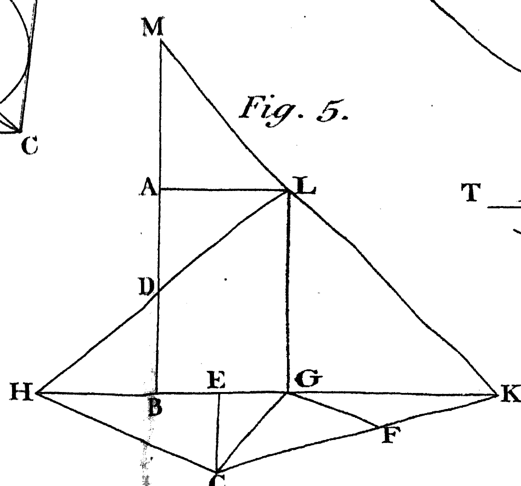


Fig. 191.

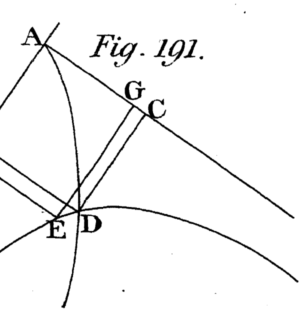
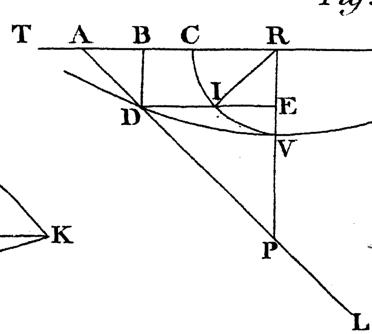


Fig.



SECTION II.

Of the Ciſſoid.

LEMMA.

Let E G F H be a circle of which C is the center, and E F, H G two diameters at right angles to one another. Let E F be produced to P, ſo that F P may be equal to C F; and A being any point in C G draw A P, and upon A P as a diameter deſcribe the ſemicircle A B P. Then if from A a ſtraight line A B, equal to C P, be inſcribed in the ſemicircle, the point in which A B is biſected will be on the ſame ſide of C P with the point A. Fig. 8.

For let K be the point in which A P is biſected, and draw K F; and draw K D perpendicular to A B. Then as A C P is a right angle, the circumference A B P (31. iii.) paſſes through C; and as A B, C P are equal, the circumference A C B (28. iii.) is equal to the circumference P B C; and therefore the circumference A C is equal to the circumference B P. The angle A P C (27. iii.) is therefore equal to the angle P A B; that is, the angle K P F is equal to the angle K A D. Again, as P K is equal to K A, and P F equal to F C, P K : K A : : P F : F C; and therefore (2. vi.) K F is parallel to C A, and (29. i.) the angle P F K is a right angle. The triangles P K F, A K D are therefore equiangular, and (26. i.) K F is equal to K D, and A D to P F. Conſequently if with K as a center and K F as a diſtance a circle be deſcribed, it will paſs through D, and C P will be a tangent to this circle (16. iii.) as it is at right angles to K F. The ſtraight line A B is alſo biſected in D, for

SECT. II.

for A D is equal to P F. Consequently the point in which A B is bisected is on the same side of C P with the point A.

DEFINITIONS.

I.

Fig. 9.

Let E G F H be a circle, of which C is the center, and E F, H G two diameters at right angles to one another. Let E F be produced to P, so that F P may be equal to C F the radius, and let H G be produced indefinitely on one side towards A a point in H G. Let the straight line B L, of indefinite length towards L, pass through P; and let B A, equal to E F or C P, be at right angles to B L, and let D be the point in which B A is bisected. Then if the straight lines B L, B A be so moved that B L always pass through P, and the extremity A of B A be always in H G, the point D will describe a curve F G D which is called a *Cissoid*, or the *Cissoid of Diocles*.

II.

The circle E G F H is called the *Generating Circle* of the cissoid; and the point P is called its *Pole*.

III.

The straight line H G A is called the *Directrix* of the cissoid.

IV.

The diameter E F of the generating circle is called the *Axis* of the cissoid, and the point F is called its *Cusp*.

V.

The straight line E K perpendicular to the axis is called the *Asymptote* to the cissoid.

VI.

A straight line drawn from any point in the cissoid perpendicular to the axis is called an *Ordinate* to the axis,

axis, and the segment of the axis between the cusp and an ordinate is called an *Abscifs* of the axis. SECT. II.

PROP. I.

The cissoid commences at the cusp, and the curve is entirely on one side of the axis; it also passes through the point in which the directrix cuts the generating circle, and being continually produced it approaches nearer and nearer to the asymptote, but never meets it.

For, the rest remaining as in the Definitions, draw the straight line P A. Then as P C A, P B A are right angles, a semicircle described upon P A as a diameter will (31. iii.) pass through B and C; and the straight line A B, equal to C P, will be inscribed in this semicircle, in every situation of A, regulated according to the first Definition. Consequently when A coincides with C the straight line A B will coincide with C P, and the point D will coincide with F. If therefore the description of the curve be supposed to begin from this situation, the cusp F will be the point at which it commences, and as soon as A has moved from C towards G the describing point D will be removed from E F towards G, according to the Lemma prefixed to the Definitions. When the distance of A, in the directrix, from C is equal to C P or A B, then the point B will coincide with C, and the describing point D will coincide with G. Through D draw K M perpendicular to C A, and let it meet the asymptote in K. Let B P cut C A in N. Then as P C N, N B A are right angles, and (15. i.) as the angles P N C, A N B are equal, the triangles P C N, A B N are equiangular. The triangles P C N, A M D are therefore equiangular, and the angle C P N is equal to the angle M A D. But as the point A recedes from G the point N also recedes from it, and there- Fig. 9.

SECT. II.

therefore the ciſſoid being continually produced the angle P N C becomes leſs and leſs, and conſequently the angle C P N, or its equal M A D, becomes greater and greater. The perpendicular D M muſt therefore continually increaſe, as the length of D A is conſtant, and conſequently D K, the diſtance of the curve from the aſymptote, muſt continually decreaſe, as K M (34. i.) is equal to E C. The point D however can never fall into E K, as D M can never become equal to D A or E C; for if it did then P B would become parallel to C A; which is impoſſible.

Cor. From the above it is evident, that an ordinate drawn from any point in the ciſſoid to the axis will cut the generating circle.

PROP. II.

An ordinate, drawn from any point in the ciſſoid to the axis, is a third proportional to its ſegment between the generating circle and the axis, and the correſponding abſciſs.

Fig. 10. 11.

From D, any point in the ciſſoid F D G whoſe cuſp is F, let D R be drawn an ordinate to the axis F E, and let it cut the generating circle in T; then T R is to R F as R F to R D.

For, the reſt remaining as in the Definitions and preceding Propoſition, let D R cut the generating circle again in S, and draw C S. Then as A D is equal to C S, and (34. i.) D M equal to R C, and D M A, C R S right angles, the ſquares of M A, S R are (47. i.) equal; and conſeqnently M A, S R are equal. Again, as in the triangles A B N, P C N, A B, P C are equal, and as the angle B N A is equal to the angle C N P, and the angle A B N equal to the angle P C N, the ſide B N is equal to the ſide C N. By ſimilar triangles alſo D A : M A : :

N A

N A : B A, and M D : M A : : N B : B A. Consequently on account of the equals (24. and its first Cor. v.) F R : S R : : C A : E F; and F R $\times$ E F is equal to S R $\times$ C A. But (3. ii.) F R $\times$ E F is equal to E R $\times$ R F together with the square of F R; and (35. iii.) E R $\times$ R F is equal to the square of S R. Consequently F R $\times$ E F is equal to the sum of the squares of F R, S R. Again, S R $\times$ C A is equal to S R $\times$ S D, for (34. i.) R D, C M are equal, and by the above M A is equal to S R. But S R $\times$ S D (3. iii.) is equal to S R $\times$ R D together with the square of S R. By the above therefore the sum of the squares of F R, S R is equal to S R $\times$ R D together with the square of S R. Consequently S R $\times$ R D is equal to the square of F R; and therefore (3. iii.) as T R is equal to S R, (17. vi.) T R : R F : : R F : R D. SECT. II.

Cor. 1. The straight lines E R, R T, R F, R D are in continued proportion. For (Cor. 8. vi.) E R : R T : : R T : R F; and as above R T : R F : : R F : R D.

Cor. 2. The equation of the cissoid is easily deduced from the last Corollary. For put E F $= a$, F R $= x$, and R D $= y$; and then E R $= a - x$, and R T^2 = E R $\times$ R F, or R T $= \sqrt{ax - x^2}$. Consequently $\sqrt{ax-x^2} : x : : x : \frac{x^2}{\sqrt{ax - x^2}} = y$, and $\frac{x^4}{ax - x^2} = y^2$, or $\frac{x^3}{a - x} = y^2$; and therefor $x^3 = ay^2 - xy^2$, which is the equation of the curve.

PROP. III.

If from the cusp of a cissoid a straight line be drawn cutting the cissoid and the generating circle, straight lines drawn through the points of section, and perpendicular to the axis, will be equally distant from the directrix.

Let F D G L be a cissoid, of which F is the cusp, F E the axis, H C G the directrix, and E G F H the generating Fig. 12.

SECT. II.

ing circle. From F draw any ſtraight line F K cutting the ciſſoid in D and the circle in K, and let D B, K M be perpendicular to the axis; the ſtraight lines D B, K M are equally diſtant from H C G.

For let C be the center of the circle, and let the directrix cut the circle and ciſſoid in G. Let B D meet the circle in A, and let K M meet it again in N, and draw E K. Then, by Prop. II. A B : B F : : B F : B D; and (4. vi.) B F : B D : : M F : M K. Conſequently (11. v.) A B : B F : : M F : M K; and therefore (6. vi.) the triangles A F B, F K M are equiangular, and the angle A F B is equal to the angle M K F. The circumference A E therefore (26. iii.) is equal to the circumference N F, which is equal to the circumference F K. The circumference F A is therefore equal to the circumference E K, and conſequently (29. iii.) the ſtraight lines F A, E K are equal. But (Cor. 8. vi.) E M is a third proportional to E F, E K; and F B is a third proportional to E F, A F. Conſequently E M, F B are equal, and therefore C M is equal to C B.

Cor. 1. Hence it is evident that the arch G K is equal to the arch G A.

Cor. 2. If equal arches as G K, G A be ſet off in the circumference of the circle, on the oppoſite ſides of H G, and perpendiculars K M, A B be drawn to the diameter E F, a ſtraight line drawn from F to K will cut the perpendicular A B and the ciſſoid in the ſame point. For the ſame reaſons a ſtraight line drawn from F to A will cut the perpendicular M K and the ciſſoid in the ſame point. For it may be proved, as above, if the ſtraight line F A cut the generating circle in A and the ciſſoid in L, and if A B, L M be drawn perpendicular to the axis, that C B, C M are equal.

SCHO-

SCHOLIUM.

Diocles, the inventor, confidered the property expreffed in the laft Corollary as the primary one of the ciffoid; and he fuppofed the defcription of the curve to be effected by means of an indefinite number of points, as D and L, obtained from equal arches as G K, G A, and the interfections of the ftraight lines F K, F A with the perpendiculars A B, M L *. Sir I. Newton firft fhewed how the ciffoid might be defcribed by continued motion according to the conditions expreffed in the firft Definition in this fection †. The method will be eafily underftood if P C A, L B A be fuppofed to be two fquares of wood or metal, P being a fixed point in P C one arm of the one, and A a fixed point in B A an arm of the other; and if the adjuftment of the inftrument and its action be fuppofed to be regulated as mentioned in the firft Definition. Fig. 9.

It is evident from the Definitions, and the Propofitions and their Corollaries, that with the fame generating circle E G F H, the fame pole P, and the fame cufp F, another ciffoid F H R may be defcribed on the oppofite fide of E P to that on which F G D is defcribed. It is alfo evident that S E K, touching the generating circle in E, is the common afymptote to thefe two ciffoids, and that E F is their common axis; and it may readily be perceived that the properties proved of the one equally apply to the other.

* See the Oxford edition of Archimedes, page 138.

† See the Appendix to the Arithmetica Univerfalis.

SECTION III.

Of the Quadratrix.

DEFINITIONS.

I.

Fig. 13. Let A D E be a femicircle, and let C be the center of the circle, and A E a diameter. Let C D be at right angles to A E and meet the circumference in D, and let it be produced to S fo that D S may be equal to C D. Let a ftraight line C H of indefinite length revolve about C, begin its revolution from a coincidence with C E, and move towards D; and at the fame time that C H begins to revolve let a ftraight line F G move from a coincidence with C E towards S, and let F G be always parallel to A E, or perpendicular to C S. Let C H revolve and F G move with an uniform velocity, and let the arch E H paffed over by the revolution of C H be to the diftance C F moved over by the extremity of F G in the fame time, as the quadrantal arch E D to the radius C E; the curve B I D defcribed by the point I, in which C H, F G cut one another, is called a *Quadratrix*, or the *Quadratrix of Dinoftrates.*

II.

The circle A D E is called the *Generating Circle* of the quadratrix.

III.

The ftraight line C S is called the *Axis* of the quadratrix.

IV.

If B be the point at which the lines C H, F G commence

mence their interfection, the ftraight line C B is called the *Bafe* of the quadratrix. SECT. III.

V.

The ftraight line S T perpendicular to the axis is called the *Afymptote* to the quadratrix.

PROP. I.

The curve of the quadratrix paffes through the point in which the axis cuts the generating circle; and being continually produced it approaches nearer and nearer to the afymptote, but never meets it.

Part I. Every thing remaining as in the Definitions, as C H revolves and F G moves with an uniform velocity, and as the velocity with which C H revolves is to the velocity with which F G moves as the quadrantal arch E D to the radius C D, the arch E H is to C F as the quadrantal arch E D to the radius C D. When the revolving line C H therefore coincides with C D, the point F in F G will coincide with D, and I will alfo coincide with D. Confequently the curve of the quadratrix muft pafs through D. Fig. 13.

Part II. The reft remaining as above, let C L reprefent the revolving line after it has proceeded beyond C D from E, and let L M reprefent the fituation of the line moving parallel to A E at the fame time, and let C L, M L interfect one another in L, and let C L cut the generating circle in K. Then it is evident, from the firft Definition, that the point L is in the curve of the quadratrix; and, for the fame reafons as above, the arch E D K is to C M as the quadrantal arch E D to the radius C D. Hence it is evident, that, if the curve of the quadratrix be continually produced, the revolving line will cut off greater and greater arches of the generating circle, reckoning from the extremity E, and

SECT. III. consequently the distances of the moving line parallel to A E must become greater and greater. The distance of the moving line parallel to A E from the asymptote T S must therefore continually decrease. Consequently the curve of the quadratrix must approach nearer and nearer to the asymptote upon being continually produced, but they can never meet, for if they did then the revolving line would coincide with A E, and A E, T S would meet. But this is impossible, for, by the fifth and first Definitions, (and 28. i.) they are parallel.

Cor. Any arch E H and distance C F, passed over by the revolving line C H and moving line F G in the same time, are to one another as the quadrantal arch E D to the radius C D. Also (19. v.) the arch D H is to D F as the quadrantal arch E D to the radius C D.

PROP. II.

The base of the quadratrix is a third proportional to the quadrantal arch of the generating circle and its radius.

Fig. 14. Let B I D be a quadratrix, of which C B is the base, A D E the generating circle, and C E or C D the radius of the circle, C being its center; the quadrantal arch E H D of the generating circle is to its radius C E as C E to C B.

For let C H be a position of the revolving line, and F I a corresponding position of the line which moves parallel to A E, and let them intersect one another in I, as in the first Definition, so that I may be in the curve of the quadratrix. Let C H, F I be indefinitely near to C E, so that the arch E H of the generating circle may be indefinitely small; and let H L, I K be perpendicular to C E. Then, by the Cor. to Prop. I. the arch E H : C F : : arch E H D : C E. But it is evident that F K is a parallelogram, and therefore (34. i.) C F is equal to

K I;

K I; and as the arch E H is indefinitely ſmall, it is equal to its ſine L H. Conſequently L H : K I : : arch E H D : C E; and therefore (II. v. and 4. vi.) arch E H D : C E : : C L : C K. But when the arch E H is indefinitely ſmall, and equal to L H, C L is equal to the radius, and C K becomes equal to C B. Conſequently the quadrantal arch E D : C E : : C E : C B.

Cor. 1. As by the above and inverſion C B : C E : : C E : the quadrantal arch E D, by the Cor. to Prop. I. (and 11. v.) C B : C D : : F D : the arch D H. Alſo by the above, Cor. to Prop. I. (and 11. v.) C B : C D : : C F : the arch E H.

Cor. 2. The equation of the quadratrix may be obtained from the above in the following manner. Put C D $= a$, C B $= b$, E H $= z$, C F $= y$; and then, as in the Propoſition $b : a : : a : \frac{a^2}{b} =$ the quadrantal arch E D. Conſequently $\frac{a^2}{b} : a : : z : \frac{bz}{a} =$ C F $= y$; and $bz = ay$, the equation of the curve.

LEMMA.

A circle is equal to a right angled triangle, which has one of the ſides round the right angle equal to the radius of the circle, and the other ſide equal to the circumference.

Let A I B D E be a circle, as in Fig. 15. of which C is the center, and C I a radius, and let K M N, as in Fig. 16. be a right angled triangle, having K M one of the ſides round the right angle at M equal to C I, and the other M N equal to the circumference of the circle; the circle A I B D E is equal to the triangle K M N.

For, if it be poſſible, let the circle be greater than the triangle, and firſt by inſcribing a ſquare in it, and afterwards by a repeated biſection of circular arches,

 let

SECT. III.

let a polygon of an even number of fides be infcribed in the circle ; and let the excefs of the circle above the polygon be lefs than its excefs above the triangle. Then the polygon thus infcribed in the circle will be greater than the triangle. Let I B be a fide of this polygon, and let C L, at right angles to it, meet it in L. Then C L is lefs than K M, and as the ftraight line I B is lefs than the circular arch I B, the perimeter of the polygon is lefs than the circumference of the circle. The rectangle under C L and the perimeter of the polygon is therefore lefs than the rectangle under K M, M N. But it is evident (from I. ii. and 34. i.) that the rectangle under C L and the perimeter of the polygon is double the area of the polygon ; and the rectangle under K M, M N is double the area of the triangle K M N. The polygon is therefore lefs than the triangle K M N ; and it is alfo greater ; which is abfurd.

But, if it be poffible, let the circle be lefs than the triangle ; and firft by defcribing a fquare about the circle, and afterwards by a repeated bifection of circular arches let a polygon be defcribed about the circle, and let the excefs of the polygon above the circle be lefs than the excefs of the triangle above the circle. And that this may be done is evident from (I. x. and I. xii.) confidering that if B H, A H touch the circle in B and A and meet one another in H, then if C H be drawn cutting the circle in I, and F G touch it in I and meet B H in F and A H in G, and B I, I A be drawn, the triangle H I F is greater than the triangle B I F, and the triangle H I G is greater than the triangle A I G. For (18. iii.) H I F, H I G are right angles, and therefore (18. i.) F H is greater than F I, and G H greater than G I. But it is evident (from 36. iii.) that B F is equal to F I, and G I to G A, and therefore (I. vi.) the triangle H I F is greater than the triangle B I F, and the tri-

SECT. III.

triangle H I G is greater than the triangle A I G. Let F G be a side of the polygon described about the circle, whose excess above the circle is less than the excess of the triangle K M N above the circle, and then the polygon, of which F G is a side, is less than the triangle K M N. But as the perimeter of the circumscribed polygon is greater than the circumference of the circle, and as the rectangle under C I and the perimeter of the circumscribed polygon is equal to the double of the area of the polygon, it is evident for the same reasons as above that the circumscribed polygon is greater than the triangle K M N. The circumscribed polygon therefore is both less and greater than the triangle K M N; which is absurd. Consequently the circle A I B D E is equal to the triangle K M N.

Cor. 1. A circle is equal to a rectangle which has one of its sides equal to the radius of the circle, and the other side round the same angle equal to half the circumference.

Cor. 2. The diameter of one circle is to the diameter of another as the circumference of the first mentioned to the circumference of the other. For put D equal to the diameter of the one and C equal to its circumference, and put d equal to the diameter of the other and c equal to its circumference. Then, by the preceding Cor. (and 2. xii.) $D \times C : d \times c :: D^2 : d^2$, and $D \times C : D^2 :: d \times c : d^2$. Consequently (1. vi.) $C : D :: c : d$.

Cor. 3. If the two circles F H G B, K L M E have the common center C, and C A, C B be drawn cutting the outer circle in A, B, and the inner in D, E, the radius C B is to the radius C E as the arch A B to the arch D E. For let F G, H B be two diameters of the outer circle at right angles to one another, and let F G cut the inner circle in K, M, and H B cut it in L, E. Then Fig. 17.

SECT. III.

(33. vi.) the arch F B : the arch A B : : the angle F C B : the angle A C B : : the arch K E : the arch D E ; and therefore, by alternation, the arch F B : the arch K E : : the arch A B : the arch D E. But the arch F B is a fourth part of the circumference F H G B, and the arch K E is a fourth part of the circumference K L M E. Consequently, by the last Cor. (and 15. v.) C B : C E : : the arch A B : the arch D E.

Such sectors as A C B, D C E which have the angles at their centers equal, are called *Similar Sectors*.

PROP. III.

If from any point in the quadratrix a straight line be drawn to the center of the generating circle, and also a straight line perpendicular to the axis, and if with the center of the generating circle, and the base as a distance, a circle be described, the arch of this circle intercepted between the extremity of the base and the straight line drawn from the quadratrix to the center, will be equal to the segment of the axis between the center and perpendicular.

Fig. 19.

From any point L in the quadratrix B D L let a straight line L C be drawn to C the center of the generating circle A D E, and with C as a center and C B, the base of the quadratrix as a distance, let a circle be described; the arch B F of this circle, intercepted between B and C L, is equal to C M, the segment of the axis C D M between C the center and L M the perpendicular.

For, by Prop. II. and inversion, C B : C D : : C D : the quadrantal arch D E; and, by Cor. 1. to Prop. II. C D : quadrantal arch D E : : C M : the arch E K ; and, by Cor. 2. to the preceding Lemma, (and 15. v.) C B : C D or C E : : the quadrantal arch B G : the quadrantal arch

rch DE. Again (33. vi.) the angle ECD : the angle CK : : the quadrantal arch ED : the arch EK ; and or the same reasons the quadrantal arch BG is to the rch BF in the same proportion. Consequently (11. v.) he quadrantal arch ED : the arch EK : : the quadrantal arch BG : the arch BF; and, by alternation, he quadrantal arch ED : the quadrantal arch BG : : he arch EK : the arch BF. By the above therefore and 11. v.) the arch BF : the arch EK : : CM : the rch EK; and (14. v.) consequently the arch BF is equal to CM.

Cor. The quadrantal arch BG is equal to the radius CD. For, as above, by Prop. II. CB : CD : : CD : the quadrantal arch DE; and, by Cor. 2. to the preceding Lemma, CB : CD : : the quadrantal arch BG : the quadrantal arch ED. Consequently (11. v.) CD : the quadrantal arch ED : : the quadrantal arch BG : the quadrantal arch ED; and therefore (14. v.) CD is equal to the quadrantal arch BG.

SCHOLIUM.

If the straight line CH revolve, and the straight line FG move with the same relative velocities as stated in the first Definition, but on the side of AE opposite to that before supposed, the quadratrix LDB may be extended as represented by LDBKO in Fig. 18. and if the generating circle be completed, and DC be produced to M so that CM be equal to CS, then a straight line MN at right angles to CM will also be an asymptote to the curve. It is also evident that the curve will pass through the point K in which CM cuts the generating circle. Fig. 18.

It is much to be wished that an instrument were devised for describing the quadratrix by continued motion, as the two following important Problems could

 then

SECT. III.

then be solved geometrically. In the following solutions the possibility of describing the quadratrix is necessarily supposed.

PROP. IV. PROB. I.

From a given rectilineal angle to cut off any part required, by means of the quadratrix.

Fig. 13. Let KCE be a given rectilineal angle; it is required to cut off any part from it, by means of the quadratrix.

With C as a center, and any convenient distance CE, let a circle ADE be described. With the generating circle ADE let the quadratrix BIDL be described, and let it meet CK in L. Let CDM be the axis of the quadratrix, and let LM be perpendicular to it. Let CM be so cut in F (9. vi.) that CM may be to CF as the whole given angle KCE to the part required. Draw FG parallel to CE, and let it cut the quadratrix in I. Draw CI and let it meet the circle in H; and the angle ECH will be the part required.

For, by the Cor. to Prop. I. (and 11. v.) CM : CF : : the arch KE : the arch HE; and (33. vi.) the arch KE : the arch HE : : the angle KCE : the angle HCE.

PROP. V. PROB. II.

To find a square equal to a given circle, by means of the quadratrix.

Fig. 18. Let ADEK be a given circle; it is required to find a square equal to it, by means of the quadratrix.

Let C be the center of the circle, and with ADEK as a generating circle, and its diameter DK as an axis, let the quadratrix LDBKO be described, and let CB, in the diameter ACE, be its base. Through B draw PQ parallel to DK, and let it meet

DP

D P perpendicular to D K in P, and K Q perpendicular to D K in Q. Draw C P, C Q, and let them meet R E V parallel to D K in R and V. Draw R W, V X perpendicular to D K. Then the right angled parallelogram W R V X is equal to the given circle A D E K, and a mean proportional between the ſides W R, R V is the ſide of the ſquare required.

For with C as a center, and C B as a diſtance, let the circle G B I be deſcribed, and let it meet D K in G and I. Then, by the Cor. to Prop. III. the quadrantal arch B G is equal to C D, and the quadrantal arch B I is equal to C K. Again, by the above conſtruction, D B, W E are parallelograms, and therefore (34. i.) D P is equal to C B, and W R is equal to C E. The triangles C D P, C W R are alſo equiangular, (29. i.) and therefore (4. vi.) D P : W R : : C D : C W. Conſequently, by Cor. 2. to the Lemma in this ſection, (and 15. v.) the quadrantal arch D E is equal to C W, and for the ſame reaſons the quadrantal arch K E is equal to C X; and therefore W X is equal to half the circumference of the circle A D E K. The right angled parallelogram W R V X is therefore, by Cor. 1. to the Lemma in this ſection, equal to the circle A D E K; and a mean proportional (13. vi.) being found, it will be equal to the ſide of the ſquare required.

Cor. It is evident, from the Cor. to Prop. III. and Cor. 1. to the Lemma in this Section, that the right angled parallelogram D P Q K is equal to the circle G B I.

SECTION IV.

Of Cycloids.

DEFINITIONS.

I.

Fig. 20. If the circle A W E roll along the ſtraight line A B ſo that every part of the circumference may touch it in regular ſucceſſion, and if at the commencement of the revolution of the circle it touch A B in A, and if during the revolution the point A remain fixed in the circumference, and at the end of it meet the ſtraight line in B; the curve A V B deſcribed by the point A, during the revolution, is called a *Cycloid*.

II.

The circle A W E is called the *Generating Circle* of the cycloid.

III.

The ſtraight line A B is called the *Baſe* of the cycloid.

Cor. As every part of the circumference of the circle touches the baſe, in regular ſucceſſion, the baſe of the cycloid is equal to the circumference of the generating circle.

IV.

The ſtraight line H V, biſecting the baſe A B at right angles and meeting the curve in V is called the *Axis* of the cycloid.

V.

The point V is called the *Vertex* of the cycloid.

VI.

A ſtraight line drawn from any point in the curve per-

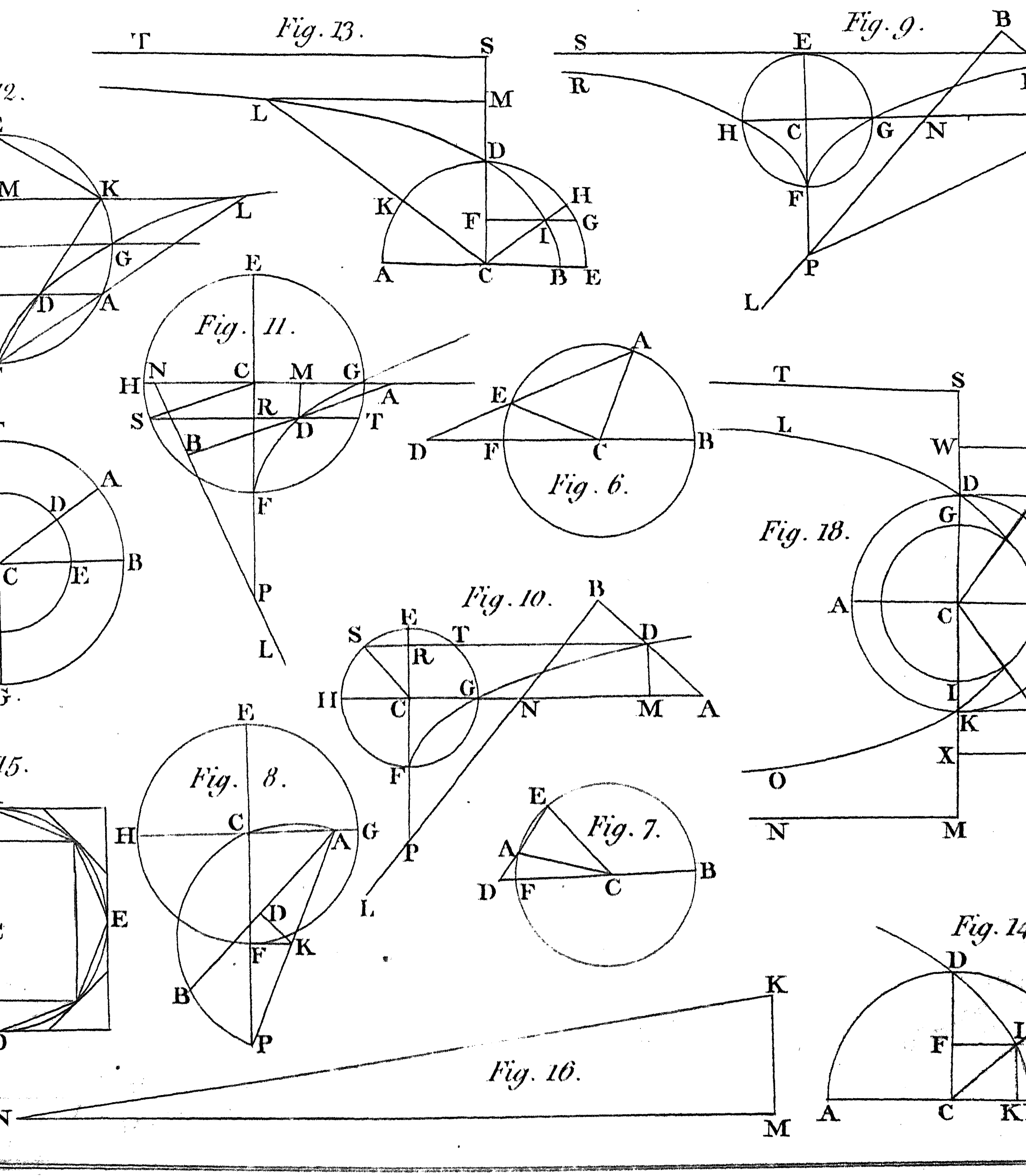


Fig. 13.
Fig. 9.
Fig. 11.
Fig. 6.
Fig. 18.
Fig. 10.
Fig. 8.
Fig. 7.
Fig. 14
Fig. 16.

perdendicular to the axis is called an *Ordinate* to the axis; and the ſegment of the axis between the vertex and an ordinate is called an *Abſciſs*. SECT. IV.

PROP. I.

During the revolution of the generating circle its center deſcribes a ſtraight line equal to the baſe of the cycloid; and the axis of the cycloid is equal to the diameter of the generating circle

For, the reſt being as in the Definitions, let C be the center of the generating circle, and A E the diameter paſſing through A. Then as the circle A W E always touches the ſtraight line A B, the center C is at the ſame diſtance from A B throughout the revolution, and therefore it muſt deſcribe a ſtraight line. At the commencement of the revolution, and alſo at the end of it, A C (18. iii.) is perpendicular to A B, and therefore it is evident that if theſe perpendiculars and the line deſcribed by the center were drawn, a parallelogram would be formed, of which the line deſcribed by the center and A B would be oppoſite ſides. The ſtraight line deſcribed by the center is therefore equal to the baſe of the cycloid. Laſtly, let A W E be the poſition of the generating circle at the beginning of the revolution, and then it is evident, from the firſt five Definitions, that at the middle of the revolution the point E will coincide with H and the deſcribing point A with V. The axis H V therefore of the cycloid is equal to the diameter of the generating circle. Fig. 20.

Cor. A circle deſcribed on the axis of a cycloid is equal to the generating circle.

PROP.

SECT. IV.

PROP. II.

An ordinate drawn from any point in the cycloid to the axis is equal to the arch of the generating circle, described on the axis, between the vertex and ordinate, together with the sine of the same arch.

Fig. 20. From any point F in the cycloid A V F B let F M be drawn an ordinate to H V the axis, and let it cut the circle V P H described on the axis in the point P; the ordinate F M is equal to the arch V P, between V the vertex and the ordinate, together with P M the sine of the same arch.

For let S F L I represent the generating circle when the describing point coincides with F, and in this situation let T denote its center, and let L be the point in which it touches A B. Draw the diameters L T S, F T I, and let L T S meet F M in N. Then S L (18. iii.) is perpendicular to A B; and, as F M is perpendicular to V H, M L is a parallelogram, and therefore (34. i.) M H is equal to N L, and M N is equal to H L. But, from the Cor. to the third Definition, the semicircle I L F is equal to H B half the base, and from the description of the curve the circular arch L F is equal to the remaining part L B of the base. Consequently H L is equal to the arch I L, or to its equal (26. iii.) S F; and as H M, L N are equal, it is evident (from Cor. 8. vi.) that M P is equal to N F, and the arch V P equal to the arch S F. The arch V P is therefore equal to M N; and as N F is equal to P M, the ordinate F M is equal to the arch V P together with P M the sine of the same arch.

Cor. The equation of the cycloid is immediately obtained from the above. For put F M $= y$, the arch V P $= z$, and P M $= s$, and then $z + s = y$, the equation of the curve.

DE-

DEFINITIONS.

VII.

Let A W E be a circle in which C is the center, A E a fixed diameter, and D a point in A E or in A E produced; and let the circle A W E roll along the ſtraight line A B ſo that every part of the circumference may touch it in regular ſucceſſion from the beginning of the revolution at A to the end of it at B; the curve D V F G deſcribed by the point D during the revolution is called a *Curtate Cycloid* if the point D be without the circle as in Fig. 21. but if D be within the circle as in Fig. 24. the curve D V F G is called a *Prolate or Inflected Cycloid*. Fig. 21. 24.

VIII.

The circle A W E is called the *Generating Circle* either of the Curtate or Prolate Cycloid.

IX.

The ſtraight line D G joining the points in which the generation of the curve begins and ends is called the *Baſe* of either of the two Cycloids.

X.

The ſtraight line R V biſecting the baſe D G at right angles and meeting the curve in V is called the *Axis*, and the point V is called the *Vertex* of either of the two cycloids.

XI.

A ſtraight line drawn from any point in the curve perpendicular to the axis is called an *Ordinate* to the axis; and the ſegment of the axis between the vertex and an ordinate is called an *Abſciſs*.

PROP. III.

In either the curtate or prolate cycloid the baſe is equal to the circumference of the generating circle; and in the curtate

SECT. IV.

curtate cycloid the axis is greater, but in the prolate it is leſs, than the diameter of the generating circle.

Fig. 21. 24.

For the reſt remaining as in the Definitions, the generating circle both at the beginning and end of the revolution touches the ſtraight line A B; and therefore D A B (18. iii.) is a right angle. And, as at the end of the revolution A D is repreſented by B G, B G is equal to A D, and A B G is a right angle. Conſequently (28. i.) A D, B G are parallel, and therefore (33. i) A B, D G are equal and parallel. But as the revolution of the generating circle begins at A and ends at B, the circumference of the circle is equal to A B, and therefore the baſe D G is equal to the circumference of the generating circle. Again, as R V biſects D G at right angles, R H is parallel to D A or G B, and therefore A B is biſected in H, and A H is equal to half the circumference of the generating circle. Conſequently in the middle of the revolution the point E will coincide with H, and the diameter E A as to poſition will coincide with the axis R V. Hence it is evident, that in the curtate cycloid the axis R V is greater than the diameter of the generating circle by the double of A D; but in the prolate cycloid the axis R V is leſs than the diameter of the generating circle by the double of A D.

Cor. From the above it is evident that in the middle of the revolution the center of the generating circle biſects the axis R V; and it is alſo evident that during the revolution the center of the generating circle is at the ſame diſtance from the baſe D G.

PROP. IV.

If an ordinate be drawn from any point in the curtate or prolate

prolate cycloid to the axis and cut a circle described on the axis as a diameter, the arch of the circle between the vertex and ordinate will be to the segment of the ordinate between the cycloid and circle as the circumference of the circle to the base of the cycloid. SECT. IV.

Let D V G be a curtate or prolate cycloid, of which D G is the base, R V the axis, and V the vertex, and from any point F in the cycloid let F M be drawn an ordinate to the axis, and let it cut the circle V P R, described upon V R as a diameter, in P; the arch V P is to the segment F P as the circumference of the circle V P R to the base D G. Fig. 21. 24.

For, the rest remaining as in the Definitions, let T be the center of the generating circle when the describing point D coincides with F. Through T draw the straight line S L Q perpendicular to D G, and let it meet A B in L. Then, as A B, D G are parallel, T L is perpendicular to A B; and, by the Cor. to the preceding Prop. and the description of the curve, T L is equal to the radius of the generating circle, and T Q is equal to the radius of the circle V P R. Through the points F and T draw the straight line F K I. With T as a center and T Q as a radius let the circle S F Q K be described; and with the same center and T L as a radius let the arch L I be described. Then, for the same reasons as in the demonstration of the second Proposition, M L is a parallelogram, P F is equal to H L, and the arch V P is equal to the arch S F. But the arch S F (26. iii.) is equal to the arch Q K; and as T L is equal to C A, from the generation of either cycloid, the arch L I is equal to the straight line H L. Again, as T Q K, T L I are similar sectors, by Cor. 2. and 3. to the Lemma in the preceding Section, (and 11. v.) the arch K Q : the arch L I : : circumference of the circle S F Q K : the

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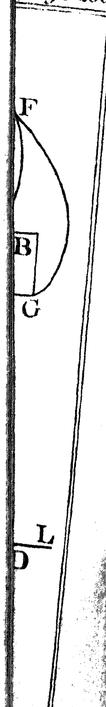


SECT. IV.

the circumference of the circle L I. Confequently, on account of the equals, the arch V P : the fegment F P : : the circumference of the circle V P R : the bafe D G.

Cor. The equation of the curtate cycloid, and alfo that of the prolate cycloid, is obtained from the above. For put $a =$ the circumference of the circle V P R, $b =$ the bafe D G, $z =$ the arch V P, $s =$ P M, and $y =$ F M; and then $a : b :: z : \frac{bz}{a} =$ P F. Confequently $y = \frac{bz}{a} + s$, the equation of the curtate and alfo that of the prolate cycloid.

SECTION V.

Of the Logarithmic Curve.

DEFINITIONS.

I.

If in the ſtraight line X Y, of an unlimited length, segments A B, B C, C D &c. be taken equal to one another, but indefinitely ſmall, and if from the points of ſection perpendiculars A E, B F, C G, D H &c. be drawn, and be in geometrical progreſſion; the perpendiculars will be indefinitely near to one another, and the line drawn through their extremities E, F, G, H, &c. is called the *Logarithmic Curve.* Fig. 22.

II.

The ſtraight line X Y is called the *Axis* to the logarithmic curve, and the perpendiculars A E, B F, C G, D H, &c. are called *Ordinates* to it.

PROPOSITION.

The axis is an aſymptote to the logarithmic curve.

For, the reſt remaining as in the Definitions, let the ordinates B F, C G, D H, &c. on the right of A E continually increaſe; and, the ſegments A *b*, *b c*, *c d*, &c. in the axis being equal to one another, and each equal to A B, let the ordinates *bf*, *c g*, *d h*, &c. on the left of A E continually decreaſe; that is, let B F be to A E as A E to *bf*, and let A E, *bf*, *c g*, *d h*, &c. be in geometrical progreſſion. Then ending the firſt rank, and beginning the ſecond with ordinates equally diſtant from A E, we have the two following ranks of magnitudes proportional taken two and two in the ſame order. Fig. 22.

SECT. V.

$$\begin{array}{ccccccc} AE & : & BF & : & CG & : & DH \\ db & : & cg & : & bf & : & AE, \end{array}$$

and therefore (22. v.) AE : DH : : db : AE; or DH : AE : : AE : db. Hence it is evident that the rectangle under any two ordinates equally diſtant from AE is equal to the ſquare of AE; and therefore if an ordinate on the right of AE be indefinitely great, an ordinate on the left of AE, and equally diſtant from it, will be indefinitely ſmall, but it can never become equal to nothing, or vaniſh. Conſequently the axis XY is an aſymptote.

SCHOLIUM.

As AB, AC, AD, &c. conſtitute a ſeries in arithmetical progreſſion, and AE, BF, CG, DH, &c. a correſponding ſeries in geometrical progreſſion, the ſegments AB, AC, AD, &c. are analogous to a ſeries of natural numbers, and the ordinates BF, CG, DH, &c. are analogous to the logarithms of theſe numbers. The curve is named from theſe analogies.

The equation of the curve is deduced from the firſt Definition, in the following manner. Put AE $= 1$, BF $= a$, and then $1 : a : : a : a^2 =$ CG; and $a : a^2 : : a^2 : a^3 =$ DH, &c. Hence it is evident that if x denote any number of equal parts AB, BC, CD in the axis, then will a^x be equal to the ordinate drawn through the extremity of the ſegment in the axis denoted by x; and therefore if this ordinate be put equal to y, then $a^x = y$, which is the equation of the curve.

The logarithmic curve, on account of its equation, is alſo called an *Exponential Curve.*

OF

OF SPIRALS.

SECTION VI.

Of the Logarithmic Spiral.

DEFINITIONS.

I.

f any number of ſtraight lines C A, C B, C D, C E, &c. Fig. 23.
wn from the point C within the curve A B D E con-
ı equal angles with it, the curve is called the *Loga-
'mic Spiral.*

II.

The point C is called the *Center* of the Logarithmic
al; and any ſtraight line drawn from the center to
curve is called an *Ordinate.*

PROPOSITION.

*' number of ordinates of the logarithmic curve are in
eometrical progreſſion, if they contain equal angles at
he center.*

.et A B D E be a logarithmic curve, of which C is Fig. 23.
center and C A, C B, C D, C E, &c. ordinates, and let
angles A C B, B C D, D C E, &c. be equal to one
:her; the ordinates C A, C B, C D, C E, &c. are in
metrical progreſſion.

or let the angles A C B, B C D, D C E, &c. be indefi-
ly ſmall, and then the portions A B, B D, D E, &c.
ıe curve may be conſidered as ſtraight lines, and as,

 by

SECT. VI.

by the firſt Definition, the angles C A B, C B D, C D E, &c. are equal, the triangles A C B, BCD, D C E, &c. are equiangular. Conſequently (4. vi.) A C : B C : : B C : D C; and B C : D C : : D C : E C. The Propoſition is therefore evident.

Cor. 1. If the ordinates B C, D C, E C, &c. on the right of A C ſucceſſively increaſe, the ordinates *b* C, *d* C, *e* C, &c. on the left of A C will ſucceſſively decreaſe. For the reſt remaining as above, if the angles A C B, A C *b*, *b* C *d*, *d* C *e*, be equal to one another, it may be proved as above that B C : A C : : A C : *b* C, and A C : *b* C : : *b* C : *d* C, &c.

Cor. 2. It may be proved, as in the logarithmic curve, that A C is a mean proportional between ordinates equally diſtant from it; and therefore as the logarithmic curve cannot meet its aſymptote at any aſſigned diſtance, ſo the logarithmic ſpiral cannot fall into its center at any aſſigned number of revolutions.

SCHOLIUM.

As the angles A C B, A C D, A C E, &c. conſtitute a ſeries in arithmetical progreſſion, and the ordinates A C, B C, D C, E C, &c. a ſeries in geometrical progreſſion, the ordinates are analogous to natural numbers, and the angles to their logarithms; and from this conſideration the ſpiral receives its name.

If A C be put $= 1$, B C $= a$, $x =$ any angle in the arithmetical ſeries, and $y =$ the correſponding ordinate, then $a^x = y$, the equation of the logarithmic ſpiral, for the ſame reaſons as ſtated in the ſcholium on the logarithmic curve.

SECTION VII.

Of the Spiral of Archimedes.

DEFINITIONS.

I.

If the ſtraight line C L revolve in a plane about one of its extremities C as a center, with an uniform velocity, and if at the commencement of the revolution a point begin to move from C and proceed in C L towards L with an uniform velocity, the line deſcribed by the point moving in C L is called *the Spiral of Archimedes.* Fig. 25.

II.

The line C G A deſcribed in the firſt revolution of C L is called *the Firſt Spiral*; the line A H B deſcribed in the ſecond revolution of C L is called *the Second Spiral*; the line B K D deſcribed in the third revolution of C L is called *the Third Spiral*, &c.

III.

If A be a fixed point in C L, and C A be the diſtance moved over by the deſcribing point during the firſt revolution, the circle deſcribed by the point A during the revolution is called *the Generating Circle*; and the point C is called the *Center* of any one of the ſpirals.

IV.

A ſtraight line drawn from C to the point in which any one of the ſpirals ends is called *the Axis* of that ſpiral; and a ſtraight line drawn from C to any other point in the ſpiral is called *an Ordinate.*

SECT. VII.

PROP. I.

If an ordinate be drawn to any point in the first spiral, and be produced till it cut the generating circle, the ordinate will be a fourth proportional to the circumference of the generating circle, the axis, and the circular arch generated from the beginning of the revolution to the point of section.

Fig. 26. Let A G be an ordinate to the first spiral A G B, of which B K H Z is the generating circle, and A being the center, let A B be the axis, and let A G be produced and cut the circle in H; the ordinate A G is a fourth proportional to the circumference B K H Z, the axis A B, and the circular arch B K H, generated from the beginning of the revolution to H the point of section.

For the line in which the fixed point B is situated revolves about A with an uniform velocity, and the point describing the spiral moves from A towards B with an uniform velocity, according to the third and first Definitions. The circumference of the circle therefore and the axis A B are described by uniform velocities in the same time; and for the same reasons the arch B K H and the ordinate A G are described by the same uniform velocities in the same time. But spaces described by the same uniform velocity are to one another as the times in which they are described, and therefore the circumference of the circle B K H Z is to the arch B K H as the time of one revolution to the time of describing the arch B K H, and A B is to A G as the same portions of time are to one another. Consequently (11. v. and alternation) the circumference of the circle B K H Z is to the axis A B as the arch B K H to the ordinate A G.

Cor. 1. The rest remaining, if A F another ordinate be produced and cut the generating circle in Z, by the above

Cor. 2. If the axis of the firſt ſpiral, or, which is the ſame thing, the radius of the generating circle, be put equal to r, the circumference of the generating circle equal to c, any arch B K H from the axis equal to z, and the correſponding ordinate A G equal to y, then $c : r :: z : \frac{rz}{c} = y$, the equation of the firſt ſpiral.

PROP. II.

An ordinate drawn to any point in the ſecond ſpiral is a fourth proportional to the circumference of the generating circle, the radius of the circle, and the ſum of the circumference of the circle and its arch generated from the beginning of the revolution to the ordinate.

Let A E be an ordinate drawn to any point E in the ſecond ſpiral B E D M, and let it cut the generating circle B K H Z in H. Let A be the center and A M the axis, and conſequently the ſituation from which the revolution begins. The circumference B K H Z of the generating circle is to its radius A B as the ſum of the circumference B K H Z and the arch B K H to the ordinate A E. Fig. 26.

For, as A L revolves with an uniform velocity, the circumference B K H Z is to the ſum of the circumference B K H Z and the arch B K H as the time of one revolution to the time in which the circumference and the arch B K H are generated by B the extremity of the radius. And theſe portions of time are to one another as the radius A B to the ordinate A E, as the point moving in A L with an uniform velocity paſſes over A B, A E in theſe portions of time reſpectively. Conſequently (II. V. and alternation) the circumference B K H Z : the

SECT. VII.

the radius A B : : the circumference B K H Z + the arch B K H : the ordinate A E.

Cor. 1. The reſt remaining as above, if A D another ordinate to the ſpiral B E D M cut the generating circle in Z, then, by the above, (and 11. v.) the circumference B K H Z + the arch B K H : A E : : the circumference B K H Z + the arch B K Z : A D.

Cor. 2. Put c = the circumference of the generating circle, r = its radius, y = the ordinate A E, and z = the arch B K H; and then, by the Propoſition, $c + z : y : : c : r$, and therefore the equation of the ſecond ſpiral is $r \times \overline{c + z} = cy$.

Cor. 3. If c denote the circumference of the generating circle, r its radius, y an ordinate drawn to any point in the nth ſpiral, and z the arch of the generating circle generated from the beginning of the revolution to the ordinate, it may be proved, as above, that $c : r : : \overline{n - 1} \times c + z : y$. Conſequently $r \times \overline{nc - c + z} = cy$ is a general equation for any one of the ſpirals of Archimedes.

Cor. 4. The reſt of the notation remaining as above, let an ordinate v contain an angle with the ordinate y, and let d denote the arch in the generating circle which meaſures this angle. Let Z denote an arch of the generating circle, generated from the beginning of the revolution to the ordinate Y; and let the ordinate Y and the ordinate x contain an angle equal to the angle contained by y and v, and conſequently alſo meaſured by an arch d in the generating circle. Then by the laſt Cor. (and 11. v.) $\overline{n - 1} \times c + z : y : : \overline{n - 1} \times c + z + d : v : : \overline{n - 1} \times c + Z : Y : : \overline{n - 1} \times c + Z + d : x$; and therefore, by Lemma X. page 154. (and 11. v.) $d : v - y : : d : x - Y$. Conſequently (14. v.) the

the exceſs of v above y is equal to the exceſs of x above Y.

Cor. 5. From the laſt Cor. it is evident, that if any number of ordinates contain angles which conſtitute a ſeries in arithmetical progreſſion, the ordinates themſelves will be in arithmetical progreſſion.

SECTION VIII.

Of the Hyperbolic or Reciprocal Spiral.

DEFINITIONS.

I.

Fig. 27. If with C, one of the extremities of the ſtraight line C L, as a center and diſtances C A, C B, C D, &c. in C L, arches of circles A G, B H, D I, &c. be deſcribed, and if theſe arches be equal to one another, and indefinitely near; the curve G H I, &c. drawn through their extremities is called *the Hyperbolic or Reciprocal Spiral.*

II.

The ſtraight line C L is called the *Axis* of the hyperbolic ſpiral, the point C its *Center*, and any ſtraight line C K drawn from the center to the curve is called an *Ordinate.*

PROP. I.

In the hyperbolic ſpiral any two ordinates are reciprocally to one another as the angles they contain with the axis.

Fig. 28. Let E G I be a hyperbolic ſpiral, of which C is the center, C L the axis, and C I, C G any two ordinates; C I is to C G as the angle L C G to the angle L C I.

For let the circular arches G A, I D be deſcribed, and let them meet the axis in A and D. Let C G produced meet the arch D I produced in M; and then as D C M, A C G are ſimilar ſectors, by Cor. 3. to the Lemma in the third ſection, C D : C A : : the arch D M : the arch A G. But C I is equal to C D, and C G to C A,

and

and by the firſt Definition the arch A G is equal to the arch D I; and therefore C I : C G : : the arch D M : the arch D I. Again (33. vi.) the arch D M : the arch D I : : the angle D C M or L C G : the angle D C I or L C I. Conſequently (11. v.) C I is to C G as the angle L C G to the angle L C I. SECT. VIII.

Cor. Every thing remaining as above, let C I be a given ordinate, and put it equal to r; put the arch D I $= a$, C G $= y$, and the arch D M $= z$. Then, by the above, $r : y : : z : a$, and $ar = yz$, the equation of the curve.

PROP. II.

If from the center of an hyperbolic ſpiral a ſtraight line be drawn at right angles to the axis and equal to one of the circular arches intercepted between the curve and axis, a ſtraight line drawn through the extremity of this perpendicular parallel to the axis will be an aſymptote to the curve.

Let E G I be an hyperbolic ſpiral, of which C is the center and C L the axis, and let the ſtraight line C B at right angles to C L be equal to I D any circular arch intercepted between the curve and the axis; the ſtraight line B F drawn through B parallel to C L is an aſymptote to the curve. Fig. 28.

For through I draw N F parallel to C B, or perpendicular to C L, and let it meet B F in F; and then N F (34. i.) is equal to C B, and therefore equal to the arch D I. But the arch D I is greater than its ſine N I, and therefore the point F in B F is without the curve. Again, from the preceding Propoſition it is manifeſt, that if the ordinate C I be indefinitely great when compared to another ordinate C G, the angle D C I is indefinitely ſmall when compared to A C G; and when an angle

SECT. VIII.

angle is indefinitely ſmall, the exceſs of the arch which meaſures it above its ſine is leſs than any aſſigned magnitude. Conſequently as I F is the exceſs of the arch D I above its ſine N I, if the curve E G I be continued till the angle D C I become indefinitely ſmall, the diſtance I F of the curve from B F will be leſs than any aſſigned magnitude. The ſtraight line B F is therefore an aſymptote to the curve.

THE END.

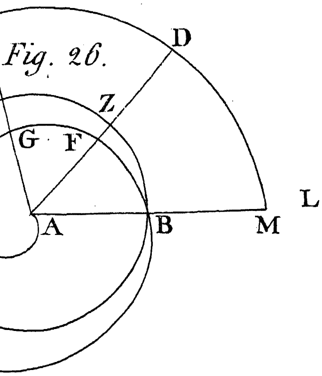
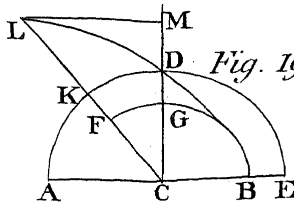


Fig. 26.



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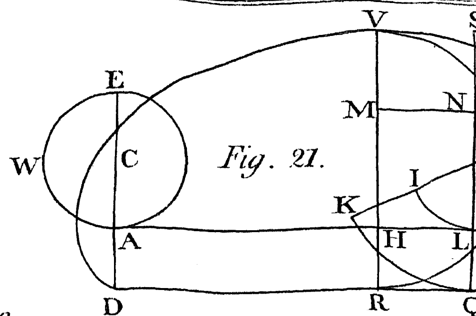
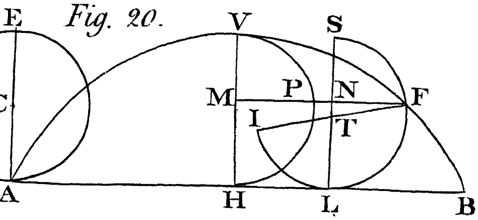


Fig. 21.



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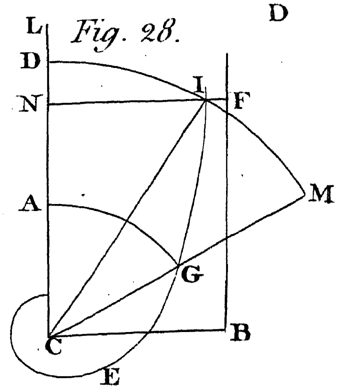

Fig. 28.

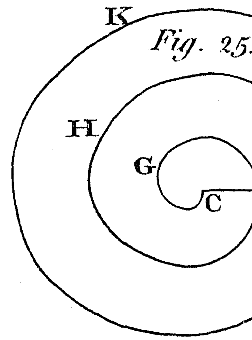

Fig. 25.

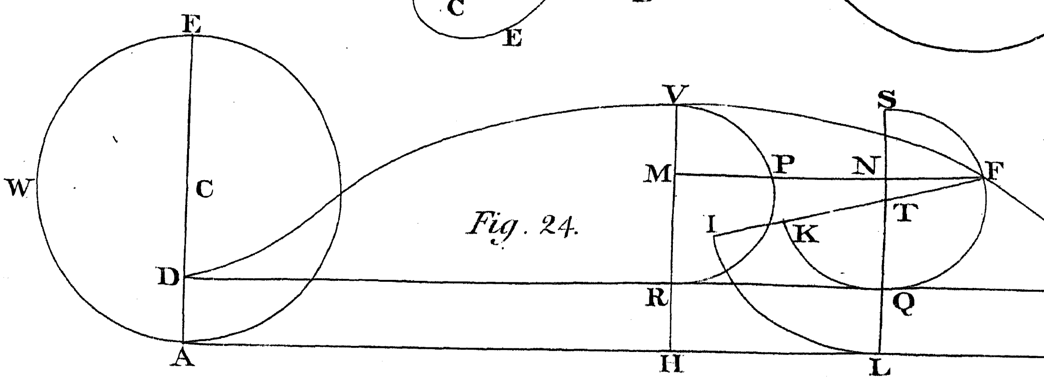

Fig. 24.

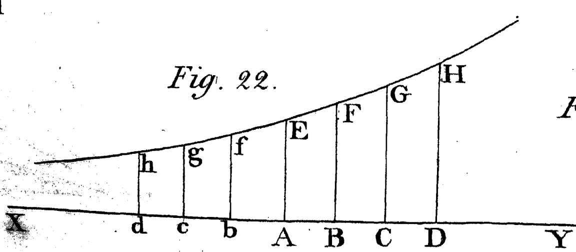

Fig. 22.

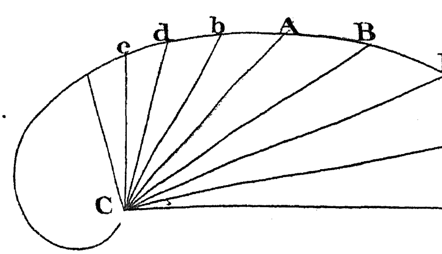

Fig. 23.